Exercise 1

8º Φροντιστήριο
ΗΥ-370

Assume an LTI system with the following transfer function: $H(z) = \dfrac{4 + 2z^{-1} - \frac{1}{2}z^{-2}}{1 - \frac{1}{4}z^{-2}}$

(a) If this system is stable, what is its impulse response?

- Long division + Partial Fraction Decomposition:

$$\begin{array}{r|l} -\frac{1}{2}z^{-2} + 2z^{-1} + 4 & -\frac{1}{4}z^{-2} + 1 \\ \hline -\left(-\frac{1}{2}z^{-2} + 0z^{-1} + 2\right) & 2 \\ \hline 2z^{-1} + 2 & \end{array}$$

$$\Rightarrow \boxed{H(z) = 2 + \frac{2z^{-1} + 2}{-\frac{1}{4}z^{-2} + 1}}$$

↳ poles: $z^{-1} = \pm 2 \Leftrightarrow z = \pm 1/2$

$$2 + \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 + \frac{1}{2}z^{-1}} = 2 + \frac{2z^{-1} + 2}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)} \Leftrightarrow$$

$$A\left(1 + \frac{1}{2}z^{-1}\right) + B\left(1 - \frac{1}{2}z^{-1}\right) = 2z^{-1} + 2$$

- $z^{-1} = -2$: $0 + B(1 - (-1)) = 2(-2) + 2 \Leftrightarrow \underline{B = -1}$

- $z^{-1} = 2$: $A(1 + 1) + 0 = 4 + 2 \Leftrightarrow \underline{A = 3}$

• In total: $\boxed{H(z) = 2 + \frac{3}{1-\frac{1}{2}z^{-1}} - \frac{1}{1+\frac{1}{2}z^{-1}}}$

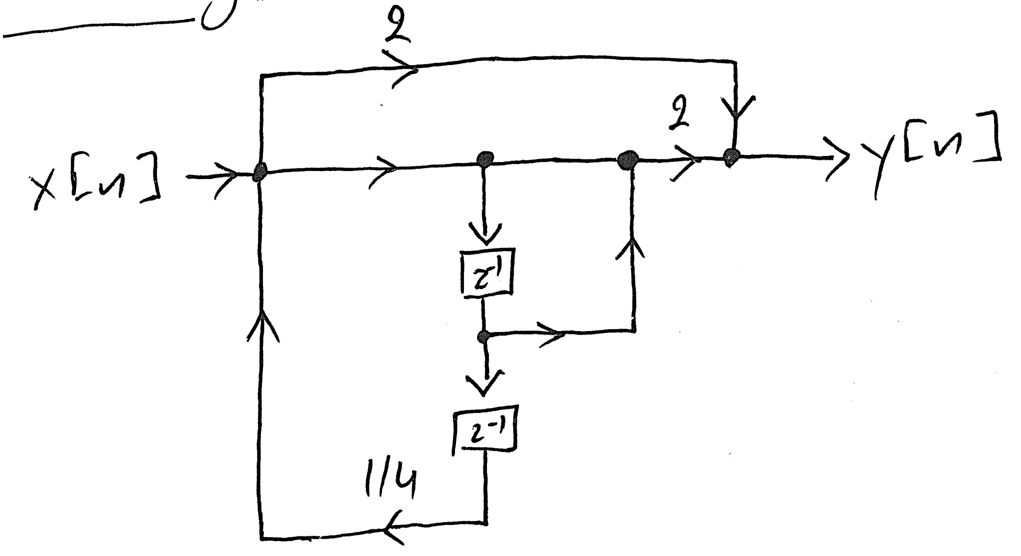
The system is assumed to be stable from the exercise, so the unit circle must be included in the ROC. Hence:

$|z| > 1/2$, and by known pairs:

$$\boxed{h[n] = \mathcal{Z}^{-1}\{H(z)\} = 2\delta[n] + 3\left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{2}\right)^n u[n]}$$

(b) Draw a graph for this system using 3 multipliers and 2 memory slots. Multiplications by ± 1 do not count.

• Chosing this form: $H(z) = 2 + 2\,\frac{1+z^{-1}}{1-\frac{1}{4}z^{-2}}$, it will be:



3 Multipliers

2 Memory Slots

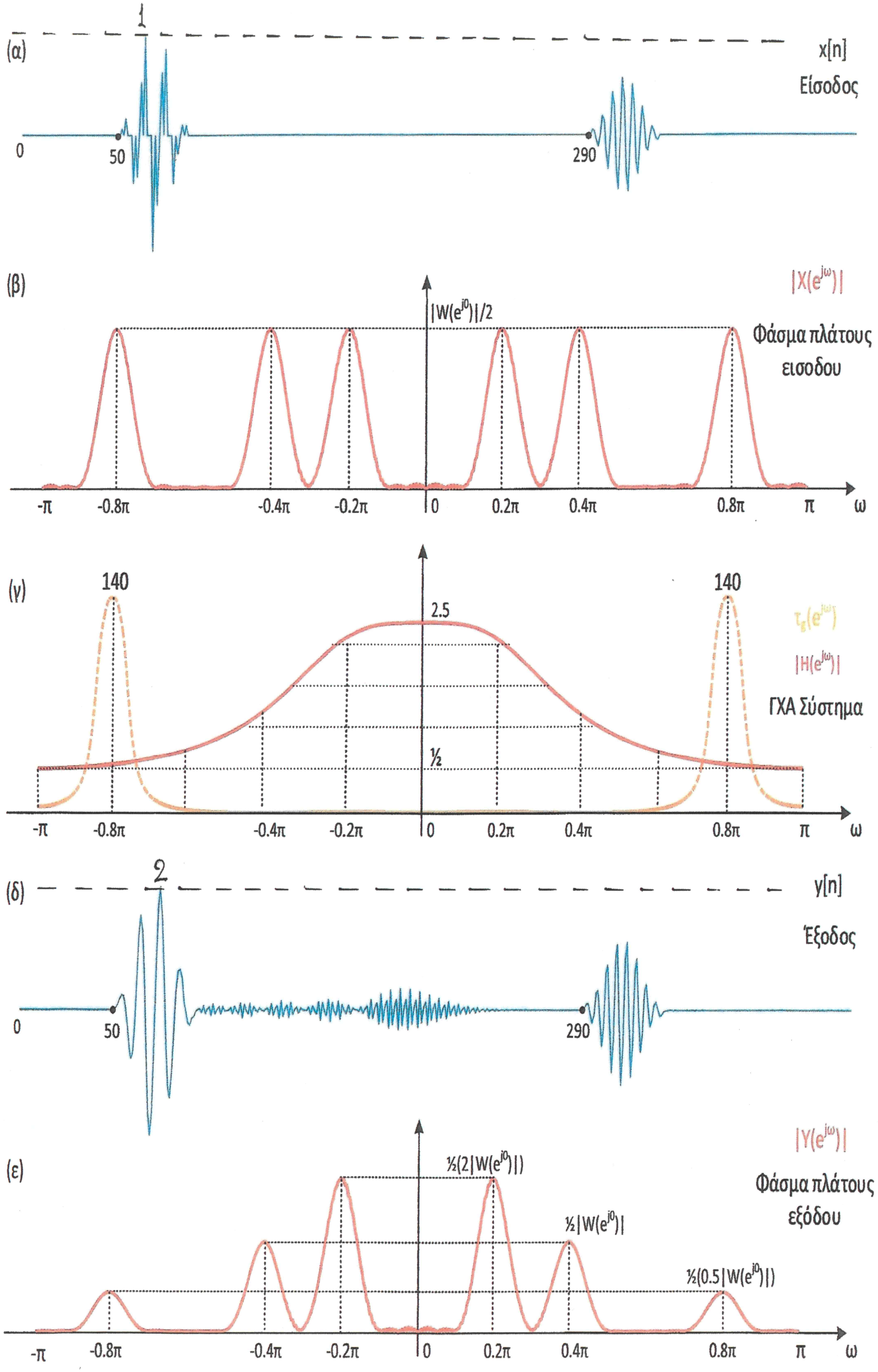
Exercise 2

In the following figure, you will see an LTI system with 1 input and 1 output. You will also see the input $x[n]$, the amplitude spectrum $|X(e^{j\omega})|$, the amplitude response $|H(e^{j\omega})|$, the system's group delay $\tau_g(e^{j\omega})$, and its amplitude spectrum $|Y(e^{j\omega})|$.

Assume the following input:

$$x[n] = w[n-50]\left(\cos(0.2n\pi) + \cos(0.8n\pi)\right) + w[n-290]\cos(0.4n\pi),$$

with $w[n]$ a window of length 41 samples, i.e., $w[n] \neq 0$, $0 \leq n \leq 40$. Explain the output behavior in this case for this system, both in time and frequency domains.



(α)
1
x[n]
Είσοδος
0
50
290
(β)
|X(e^{jω})|
|W(e^{j0})|/2
Φάσμα πλάτους
εισοδου
-π
-0.8π
-0.4π
-0.2π
0
0.2π
0.4π
0.8π
π
ω
(γ)
140
140
2.5
τ_g(e^{jω})
|H(e^{jω})|
ΓΧΑ Σύστημα
½
-π
-0.8π
-0.4π
-0.2π
0
0.2π
0.4π
0.8π
π
ω
(δ)
2
y[n]
Έξοδος
0
50
290
|Y(e^{jω})|
(ε)
½(2|W(e^{j0})|)
Φάσμα πλάτους
εξόδου
½|W(e^{j0})|
½(0.5|W(e^{j0})|)
-π
-0.8π
-0.4π
-0.2π
0
0.2π
0.4π
0.8π
π
ω

- From the input equation and the $x[n]$, $|X(e^{j\omega})|$ plots, we can infer that at $n=50$, frequencies 0.2π, 0.8π are added under the same window, while at $n=250$, the other frequency 0.4π is present.

- From the $|H(e^{j\omega})|$ plot, we can see that:

  ① $\omega = 0.2\pi$ frequency will be amplified by a factor of 2.

  ② $\omega = 0.4\pi$ ——— " ——— " — 1.

  ③ $\omega = 0.8\pi$ ——— " ——— " — 1/2.

  Figure (ε) confirms these observations

- From the $T_g(e^{j\omega})$ plot, we can see that only frequency $\omega = 0.8\pi$ will be delayed, while the others will remain unaffected. We can confirm that from figure (δ), where the frequency $\omega = 0.8$ has been moved away from its original position, while the other two have stayed at their original spot.

- To further explain its "scattering" as seen from figure (δ), it is because the lobe of $\omega = 0.8\pi$ at the $T_g(e^{j\omega})$ plot is not narrow, i.e., $\omega \in [0.7\pi, 0.9\pi]$ approximately will not be delayed by the same amount of samples, explaining, therefore, this frequency "scattering" after passing through this system.

Exercise 3

A stable LTI system has the following transfer function:

$$H(z) = \frac{\left(1-\frac{3}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}{1+z^{-1}+\frac{1}{2}z^{-1}}$$

Find a stable and causal system, such that it holds that:

$$|G(e^{j\omega})H(e^{j\omega})| = 1 \Rightarrow |H(e^{j\omega})| = \frac{1}{|G(e^{j\omega})|},$$

where $G(e^{j\omega})$ is the frequency response of that system.

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- We are essentially looking for a stable and causal inverse for $H(z)$, so $|H^{-1}(e^{j\omega})| = |G(e^{j\omega})|$ can hold. From our theory, we know that minimum phase systems that are causal and stable always have a stable and causal inverse (see also the final exercise from the previous tutorial).
- Let us find the zeroes and poles of $H(z)$ first:

poles: $1+z^{-1}+\frac{1}{2}z^{-1} = 0 \Leftrightarrow z^2+z+\frac{1}{2} = 0$, $\Delta = 1-4\cdot\frac{1}{2} = -1$, so:

$\boxed{z = \frac{-1 \pm j}{2}}$, already inside the unit circle.

zeroes: $\boxed{z = \frac{3}{2},\ z = \frac{1}{3}}$, $z = 3/2$ not in the unit circle.

- $H(z)$ is not minimum phase due to the zero at $z = 3/2$. However, we know that changing a zero to its mutual conjugate position, i.e., from $z = a$ to $z = 1/a^*$ does not change the amplitude response of the system, iff it is done like so:

$$1 - az^{-1} \longrightarrow z^{-1} - a^* \ , \quad a \in \mathbb{C}$$

(see also exercise 5 from the previous tutorial).

- Hence, $z = 3/2$ will change to $z = 2/3$ to get the minimum phase version of $H(z)$:

$$H_{min}(z) = \frac{\left(z^{-1} - \frac{3}{2}\right)\left(1 - \frac{1}{3}z^{-1}\right)}{1 + z^{-1} + \frac{1}{2}z^{-1}} = -\frac{3}{2}\,\frac{\left(z^{-1} - \frac{3}{2}\right)\left(1 - \frac{1}{3}z^{-1}\right)}{1 + z^{-1} + \frac{1}{2}z^{-1}} \ ,$$

causal and stable, so its inverse will also be:

$$\boxed{G(e^{j\omega}) = H^{-1}_{min}(z)\Big|_{z = e^{j\omega}} = -\frac{2}{3}\,\frac{1 + e^{-j\omega} + \frac{1}{2}e^{-j2\omega}}{\left(1 - \frac{2}{3}e^{-j\omega}\right)\left(1 - \frac{1}{3}e^{-j\omega}\right)}} \ ,$$

observing that: $|G(e^{j\omega})H(e^{j\omega})| = 1 \Rightarrow |H(e^{j\omega})| = \dfrac{1}{|G(e^{j\omega})|}$

holds for these systems.

Exercise 4

The following information is true about an LTI system:

(i) It has (generalized) linear phase.

(ii) It completely cuts off frequency $\omega_0 = \pm\pi/3$.

(iii) Its amplitude response is 1 for $\omega = 0$ and $\omega = \pi$.

Find the system with the minimum possible duration in samples in the time domain that satisfies all of the above. What linear phase type is it?

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- Generalized linear phase => FIR (for this course) and causal, so all poles at $z=0$, and stable too.
- Cut off $\omega_0 = \pm\pi/3$ => zeroes at $e^{\pm j\pi/3}$.
- So far: $\boxed{H(z) = A(1-e^{j\pi/3}z^{-1})(1-e^{-j\pi/3}z^{-1})}$
- Need to find the amplitude response for (iii):

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = A\left(1-e^{j\pi/3}e^{-j\omega}\right)\left(1-e^{-j\pi/3}e^{-j\omega}\right) =$$

$$= A\left(1 - e^{-j\pi/3}e^{-j\omega} - e^{j\pi/3}e^{-j\omega} + e^{j\pi/3}e^{-j\omega}e^{-j\pi/3}e^{-j\omega}\right)$$

$$= A\left(1 - e^{-j\omega}\left(e^{-j\pi/3} + e^{j\pi/3}\right) + e^{-j(-\pi/3+\omega+\pi/3+\omega)}\right)$$

$$= A\left(1 - e^{-j\omega}\, 2\cos(\pi/3) + e^{-j2\omega}\right) =$$

$$= \boxed{A\left(1 - e^{-j\omega} + e^{-j2\omega}\right) = H(e^{j\omega})}$$

• From (iii): $H(e^{j0}) = H(e^{j\pi}) = 1 \Rightarrow$

$A(1 - 1 + 1) = 1 \Leftrightarrow A = 1$, and:

$A(1 + 1 + 1) = 1 \Leftrightarrow A = 1/3$, impossible.

So, the system's equation is not complete yet. Let us add, then, another linear phase term on top: $1 + Bz^{-1} + z^{-2}$, i.e.,

$$H(z) = H(e^{j\omega})\Big|_{z = e^{j\omega}} \cdot \left(1 + Bz^{-1} + z^{-2}\right) =$$

$$= \boxed{A\left(1 - z^{-1} + z^{-2}\right)\left(1 + Bz^{-1} + z^{-2}\right) = H(z)}, \text{ or:}$$

$$\boxed{A\left(1 - e^{-j\omega} + e^{-j2\omega}\right)\left(1 + Be^{-j\omega} + e^{-j2\omega}\right) = H(e^{j\omega})}$$

Now with (iii): $H(e^{j0}) = 1 \Rightarrow A(1 - 1 + 1)(1 + B + 1) = 1 \Leftrightarrow$

$\Leftrightarrow \underline{A(B + 2) = 1}$, $H(e^{j\pi}) = 1 \Rightarrow A(1 + 1 + 1)(1 - B + 1) = 1 \Leftrightarrow$

$\Leftrightarrow \underline{3A(2 - B) = 1}$, solving the system:

$\frac{A(B+2)}{3A(2-B)} = 1 \Leftrightarrow B+2 = 3(2-B) \Leftrightarrow B+2 = 6-3B \Leftrightarrow$

$\Leftrightarrow 4B = 4 \Leftrightarrow \boxed{B=1}$, so $A(1+2) = 1 \Leftrightarrow$ $\boxed{A = 1/3}$

• In total: $H(z) = \frac{1}{3}\left(1 - z^{-1} + z^{-2}\right)\left(1 + z^{-1} + z^{-2}\right) =$

$= \frac{1}{3}\left(1 + z^{-2} - z^{-1}\right)\left(1 + z^{-2} + z^{-1}\right) = \frac{1}{3}\left(\left(1+z^{-2}\right)^2 - \left(z^{-1}\right)^2\right) =$

$= \frac{1}{3}\left(1 + 2z^{-2} + z^{-4} - z^{-2}\right) = \boxed{\frac{1}{3}\left(1 + z^{-2} + z^{-4}\right) = H(z)}$

↳ Stable and causal linear phase filter with $M=4$ zeroes, i.e., even, so either type I or III. However, it cannot be type III because $H(e^{j0}) \neq H(e^{j\pi}) \neq 0$, but 1, so it is type I.

↳ Adding more zeroes will increase its duration, so we do not progress further.

Exercise 5

The causal LTI system with transfer function:

$$H_1(z) = \frac{3}{1 - \frac{1}{2} z^{-1}},$$

is connected in parallel with the anti-causal LTI system with transfer function:

$$H_2(z) = \frac{2}{1 - 2z^{-1}}.$$

If $H(e^{j\omega})$ is the overall frequency response, calculate:

$$\int_{-\pi}^{\pi} H(e^{j\omega})\, d\omega.$$

—

- In parallel means: $H(z) = H_1(z) + H_2(z)$, or: $h[n] = h_1[n] + h_2[n]$, where $h_1[n]$ and $h_2[n]$ can be found via known properties and pairs of the z transform:

$$h[n] = Z^{-1}\left\{\frac{3}{1 - \frac{1}{2}z^{-1}}\right\} + Z^{-1}\left\{\frac{2}{1 - 2z^{-1}}\right\}$$

$$= 3\left(\frac{1}{2}\right)^n u[n] - 2(2)^n u[-n-1]$$

(we know the ROCs $|z| > 1/2$ and $|z| < 2$ correspondingly)

Now, we can calculate the given integral with the classic "trick":

$$\int_{-\pi}^{\pi} H(e^{j\omega})\,d\omega = \int_{-\pi}^{\pi} H(e^{j\omega})\,e^{j\omega 0}\,d\omega = \boxed{2\pi \cdot \mathcal{F}^{-1}\{H(e^{j\omega})\}\Big|_{n=0}}$$

$$= 2\pi\, h[n]\Big|_{n=0} = 2\pi\left(3\cdot(1/2)^0 u[0] - 2\cdot 2^0 u[-1]\right) =$$

$$= 2\pi(3\cdot 1\cdot 1 - 2\cdot 1\cdot 0) = \underline{6\pi}.$$

Exercise 6

A system is described by the following equation:

$$Y(e^{j\omega}) = e^{-j\omega}X(e^{j\omega}) + \frac{d}{d\omega}X(e^{j\omega})$$

(i) What is the system's response for input $x[n] = \delta[n]$?

(ii) Is the system stable and time-invariant?

(i) $x[n] = \delta[n] \xrightarrow{\mathcal{F}} X(e^{j\omega}) = 1$, therefore:

$$Y(e^{j\omega}) = e^{-j\omega}\cdot 1 + \frac{d}{d\omega}1 \Leftrightarrow \boxed{Y(e^{j\omega}) = e^{-j\omega}}$$

$$\xrightarrow{\mathcal{F}^{-1}} \boxed{y[n] = \delta[n-1]}$$

(ii) $Y(e^{j\omega}) = e^{-j\omega} X(e^{j\omega}) + \frac{d}{d\omega} X(e^{j\omega})$

$\mathcal{F}^{-1}$ (properties)

$$\boxed{y[n] = x[n-1] - jn\,x[n]}$$

• Time-invariance $\Leftrightarrow$ input $x[n-n_0] \Rightarrow$ output $y[n-n_0]$:

Delaying the input by n_0 samles does not yield $y[n-n_0]$

in our case:

input: $x_1[n] = x[n-n_0]$

output: $y_1[n] = x[n-n_0-1] - jn\,x[n-n_0] \neq y[n-n_0]$,

due to the term n in front of $x[n]$. So, it is not time-invariant.

• Stability (BIBO) $\Leftrightarrow$ bounded input $|x[n]| < B_x \Rightarrow$ bounded output $|y[n]| < B_y$, $B_x, B_y \in \mathbb{R}$.

Our system is not stable either:

$|y[n]| = |x[n-1] - jn\,x[n]|$, because even if $x[n]$ is bounded, it holds that:

$\lim_{n \to +\infty} |y[n]| = +\infty$, again due to the same term.

Exercise 7

Causal LTI system with transfer function:

$$H(z) = \frac{1}{1 + \sum_{n=1}^{N} a_n z^{-n}},$$

is unstable. Modify it s.t. $h'[n] = \lambda^n h[n]$, and show that picking the right λ can yield a stable system instead.

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- A polynomial of degree N will have N roots in $\mathbb{C}$, so we can refactor $H(z)$ as follows:

$$H(z) = \frac{1}{\prod_{n=1}^{N} (1 - d_n z^{-1})},$$

where d_n will be the location of the n-th pole of the system.

We know that $\mathcal{Z}\{\lambda^n h[n]\} = H(z/\lambda)$, so the new system is:

$$\boxed{H'(z) = \frac{1}{\prod_{n=1}^{N} (1 - \lambda d_n z^{-1})}}$$

- Unstable means that at least one pole is outside the unit circle, because we already know that the system is causal ($|z| > a$) from the exercise.

• If d_{max} denotes the pole with the highest magnitude for this system, then simply by scaling all the poles of the system with a scalar less than $\frac{1}{|d_{max}|}$, that will bring all the poles from outside the unit circle to inside the unit circle (reciprocal property).

• This scalar, in our case, is λ, so by picking any $\lambda < \frac{1}{|d_{max}|}$ will make the system $H'(z)$ stable. E.g., if we have a pole at $z = 2$ and a pole at $z = 1/2$, then scaling them by $\lambda < 1/2$, say, 0.49, will change their location to $z = 0.98$ and $z = 0.245$, making this causal system stable (from unstable).

If $d_i = |d_i| e^{j\theta_i}$ are the poles with $|d_i| > 1$, then in $h'[n]$ they are expressed as $\lambda \cdot d_i$ where $|\lambda \cdot d_i| > 1$, so if $\lambda < 1/|d_{max}|$, they move inside the unit circle making the causal system $H'(z)$ stable.

END