Exercise 1

7° Φροντιστήριο
HY-370

• Find the coefficients of the FIR linear phase system:

$y[n] = b_0 \cdot x[n] + b_1 \cdot x[n-1] + b_2 \cdot x[n-2]$, where:

$b_0, b_1, b_2 \neq 0$, s.t.: (i) Completely cutts off frequencies $w_0 = 2\pi/3$,

and (ii) $H(e^{j0}) = 1$ holds.

Also, calculate the amplitude & phase response:

• $\mathcal{F}\{y[n]\} = Y(e^{jw}) = b_0 X(e^{jw}) + b_1 X(e^{jw}) e^{-jw} + b_2 X(e^{jw}) e^{-2jw} \Leftrightarrow$

$\Leftrightarrow H(e^{jw}) = \dfrac{Y(e^{jw})}{X(e^{jw})} = b_0 + b_1 e^{-jw} + b_2 e^{-j2w}$

$e^{jx} = \cos(x) - j\sin(x)$
Euler

↳ from (ii): $H(e^{j0}) = 1 \Rightarrow \boxed{b_0 + b_1 + b_2 = 1}$, (I)

↳ from (i): $H(e^{j2\pi/3}) = 0 \Rightarrow b_0 + b_1 \cdot e^{-j2\pi/3} + b_2 \cdot e^{-j4\pi/3} = 0 \Leftrightarrow$

$\Leftrightarrow b_0 + b_1\left(\cos\left(\frac{2\pi}{3}\right) - j\sin\left(\frac{2\pi}{3}\right)\right) + b_2\left(\cos\left(\frac{4\pi}{3}\right) - j\sin\left(\frac{4\pi}{3}\right)\right) = 0$

$\Leftrightarrow b_0 + b_1\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) + b_2\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = 0$

$\Leftrightarrow 2b_0 - b_1 - j\sqrt{3}\, b_1 - b_2 + j\sqrt{3}\, b_2 = 0$

Both real & imaginary parts must be zero.

$$\Leftrightarrow \begin{cases} 2b_0 - b_1 - b_2 = 0 \\ -\sqrt{3}\, b_1 + \sqrt{3}\, b_2 = 0 \end{cases} \Leftrightarrow \boxed{\begin{cases} 2b_0 - b_1 - b_2 = 0, & (II) \\ b_1 = b_2, & (III) \end{cases}}$$

$$(\mathrm{I}) + (\mathrm{II}) \Rightarrow 3b_0 = 1 \Leftrightarrow b_0 = 1/3 \quad , (\mathrm{IV})$$

$$(\mathrm{I}) \overset{(\mathrm{III})}{\underset{(\mathrm{IV})}{\Rightarrow}} \frac{1}{3} + 2b_1 = 1 \Leftrightarrow b_1 = 1/3 \quad , (\mathrm{V})$$

$$(\mathrm{II}) \overset{(\mathrm{V})}{\Rightarrow} b_2 = 1/3 \quad , (\mathrm{VI})$$

Hence: $H(e^{jw}) = \frac{1}{3} + \frac{1}{3} e^{-jw} + \frac{1}{3} e^{-2jw} =$

$$= \frac{1}{3}\left(1 + e^{-jw} + e^{-2jw}\right) =$$

$$= \frac{1}{3} e^{-jw}\left(e^{jw} + 1 + e^{-jw}\right)$$

$$\cos(x) = \frac{e^{-jx} + e^{+jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$= \boxed{\frac{1}{3} e^{-jw}\left(2\cos(w) + 1\right) = H(e^{jw})}$$

So: $|H(e^{jw})| = \left|\frac{1}{3}\right| \left|e^{-jw}\right| \left|2\cos(w) + 1\right| = \frac{1}{3}\left|2\cos(w) + 1\right|,$

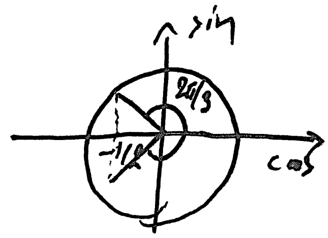
and: $\measuredangle H(e^{jw}) = \measuredangle \frac{1}{3} + \measuredangle e^{-jw} + \measuredangle (2\cos(w) + 1)$

Reminder for the phase of real functions: $T(w) \in \mathbb{R}$

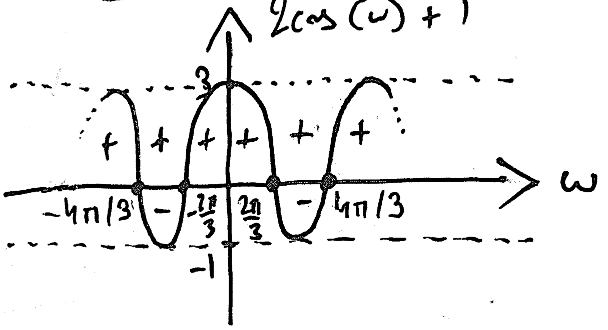
$$\measuredangle T(w) = \begin{cases} 0, & T(w) > 0 \\ \pi, & T(w) < 0,\ w > 0 \\ -\pi, & T(w) < 0,\ w < 0 \end{cases}$$

The phase is normally not defined for $T(w) = 0$.

- $2\cos(\omega) + 1 = 0 \iff \cos(\omega) = -\frac{1}{2} \iff$



$$\iff \boxed{\omega = \frac{2\pi}{3} \pm 2\pi k, \ \omega = \frac{4\pi}{3} \pm 2\pi k}$$



- phase is expressed in $[-\pi, \pi]$:

$$\measuredangle\left(2\cos(\omega)+1\right) = \begin{cases} 0, & -\frac{2\pi}{3} < \omega < \frac{2\pi}{3} \\ \pi, & \frac{2\pi}{3} < \omega < \pi \\ -\pi, & -\pi < \omega < -\frac{2\pi}{3} \end{cases}$$

- Therefore, in total:

$$\boxed{\measuredangle H(e^{j\omega}) = \begin{cases} -\omega, & -2\pi/3 < \omega < 2\pi/3 \\ -\omega+\pi, & 2\pi/3 < \omega < \pi \\ -\omega-\pi, & -\pi < \omega < -2\pi/3 \end{cases}}$$

Exercise 2

We have a linear phase filter: $H_1(z) = 1 - z^{-1}$

(i) Factorize: $H_1(e^{j\omega}) = A_1(e^{j\omega})\, e^{j\varphi_1(e^{j\omega})}$, $A_1(e^{j\omega}) \in \mathbb{R}$

- We can infer that the ROC will be the entire z-plane except $z = 0$, meaning that this is also a causal and FIR system. So, since the unit circle is included in the ROC, we can go directly to the frequency domain from the z domain:

• $H_1(e^{j\omega}) = H_1(z)\Big|_{z=e^{j\omega}} = 1 - z^{-1}\Big|_{z=e^{j\omega}} = 1 - e^{-j\omega} =$

$$= e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right) = e^{-j\omega/2} \cdot 2j\sin(\omega/2) =$$

$$= e^{-j\omega/2} \cdot 2 \cdot e^{j\pi/2} \cdot \sin(\omega/2) = e^{j\left(-\frac{\omega}{2}+\frac{\pi}{2}\right)} \cdot 2\sin(\omega/2).$$

We factorized $H_1(e^{j\omega}) = e^{j\varphi_1(e^{j\omega})} A_1(e^{j\omega})$, where:

$\boxed{\varphi_1(e^{j\omega}) = -\frac{\omega}{2} + \frac{\pi}{2}}$, $\boxed{A_1(e^{j\omega}) = 2\sin(\omega/2) \in \mathbb{R}.}$ $\Rightarrow \underline{H_1(e^{j0}) = 0}$

This is a type IV linear phase system.

(ii) $H_1(z)$ is connected in series with a generalized linear phase system $H_2(e^{j\omega}) = A_2(e^{j\omega})\, e^{-j\omega M/2}$, which is type II linear phase. Show that the overall system is type III linear phase.

• In series implies: $h[n] = h_1[n] * h_2[n]$, or

$H(e^{j\omega}) = H_1(e^{j\omega}) H_2(e^{j\omega}) = A_1(e^{j\omega}) A_2(e^{j\omega}) \cdot e^{j(\varphi_1(e^{j\omega}) + \varphi_2(e^{j\omega}))}$

in our case. Since $H_2(e^{j\omega})$ is also linear phase, then $A_2(e^{j\omega}) \in \mathbb{R}$,

hence, $A_1(e^{j\omega}) A_2(e^{j\omega}) = A_3(e^{j\omega}) \in \mathbb{R}$.

III or IV, $\underline{H_3(e^{j0}) = 0}$

• $\varphi_1(e^{j\omega}) + \varphi_2(e^{j\omega}) = -\frac{\omega}{2} + \frac{\pi}{2} - \frac{\omega M}{2} = \boxed{-\frac{\omega(M+1)}{2} + \frac{\pi}{2} = \varphi_3(e^{j\omega}).}$

$H_2(e^{j\omega})$ is type II $\Rightarrow M$ is odd $\Rightarrow M+1$ is even $\Rightarrow$ $\underline{H_3(e^{j\omega}) \text{ is type III.}}$

Exercise 3

Given the frequency response of an LTI system:

$$H(e^{jw}) = e^{-j3w}\left(1+\cos(w)+\frac{2}{5}\cos(2w)-\frac{1}{5}\cos(3w)\right)$$

(a) Classify it as FIR or IIR.

- From $\cos(x) = (e^{jx}+e^{-jx})/2$ and $\sin(x) = (e^{jx}-e^{-jx})/2j$, we can refactor $H(e^{jw})$ to be a sum of complex exponentials. In turn, we know that $e^{-jwn_0} \xrightarrow{F^{-1}} \delta[n-n_0]$, so $h[n]$ will be comprised of delta functions (not an infinite amount of them, because we have 3 sinusoids in the frequency domain). So, this system is FIR.

(b) Find the impulse response $h[n]$ for this system.

- $$H(e^{jw}) = e^{-j3w}\left(1+\frac{e^{jw}+e^{-jw}}{2}+\frac{2}{5}\cdot\frac{e^{2jw}+e^{-2jw}}{2}-\frac{1}{5}\cdot\frac{e^{3jw}+e^{-3jw}}{2}\right) =$$

$$= e^{-j3w}+0.5\left(e^{-j2w}+e^{-j4w}\right)+0.2\left(e^{-jw}+e^{-j5w}\right)-0.1\left(1+e^{-j6w}\right)$$

F^{-1}

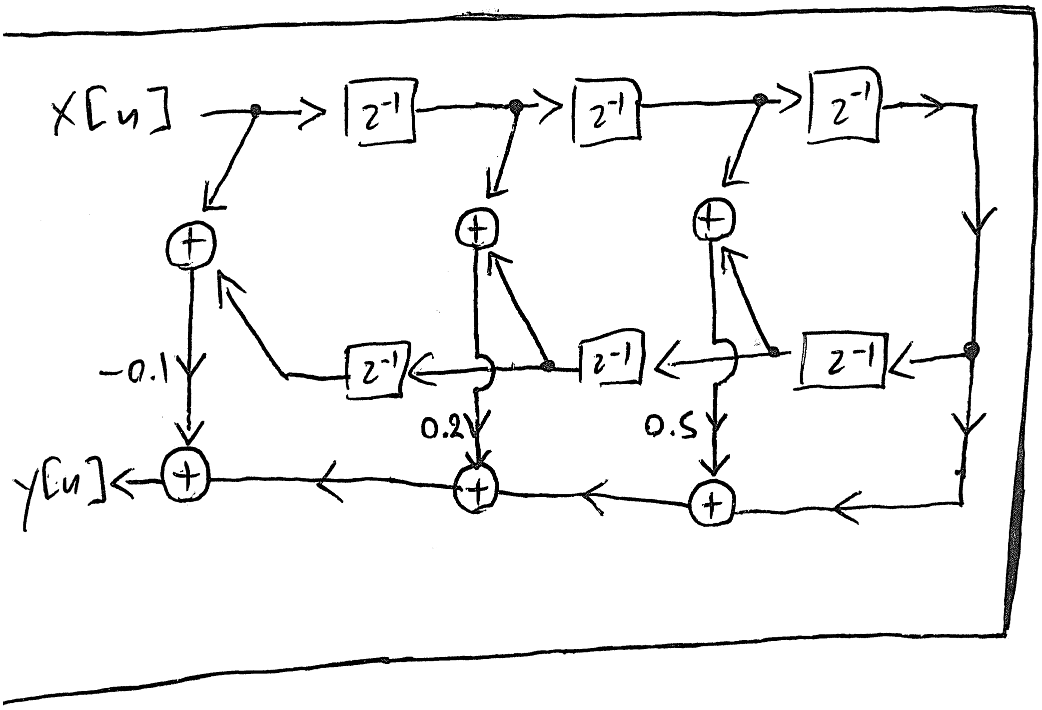
$$h[n] = \delta[n-3]+0.5(\delta[n-2]+\delta[n-4])+0.2(\delta[n-1]+\delta[n-5])$$
$$-0.1(\delta[n]+\delta[n-6])$$

$$= \boxed{-0.1\delta[n]+0.2\delta[n-1]+0.5\delta[n-2]+\delta[n-3]+0.5\delta[n-4]+ +0.2\delta[n-5]-0.1\delta[n-6] = h[n]}$$

(c) Draw a graph that represents this system.

- Observe that this is a type I linear phase system from noticing the symmetry of the delta functions from $h[n]$ that characterize type I linear phase systems (see fig. 17.39 from theory). Here, $h[n]$'s length is $M+1=7$ samples, so $M=6$, even, as expected.

- From our theory, we know how to create graphs of linear phase systems (see fig. 16.59 for type I, II). Hence, for our case:



⊕: Adders

Type I Linear Phase

$M=6$.

Exercise 4

Draw a graph of the following LTI system:

$$H(z)=\frac{16+9z^{-1}-z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}},$$

as a Parallel Form with 1-st order subsystems in Direct Form.

• We need to refactor $H(z)$ in order to draw it as Parallel Form as we know from our theory (section 16.9.2.3). We will perform long division and then decomposition into partial fractions like so:

$$
\begin{array}{r|l}
-z^{-2}+9z^{-1}+16 & -\frac{1}{8}z^{-2}+\frac{1}{4}z^{-1}+1 \\ \hline
-(z^{-2}+2z^{-1}+8) & 8 \\ \hline
7z^{-1}+8 &
\end{array}
$$

$$\Rightarrow \boxed{H(z) = 8 + \frac{8+7z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}}$$

poles: $1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}=0 \Leftrightarrow 8z^2+2z-1=0$, $\Delta = 4+32=36$,

$$z_{1,2} = \frac{-2\pm 6}{16} \quad \begin{matrix} \nearrow z_1 = -1/2 \\ \searrow z_2 = 1/4 \end{matrix}$$

, hence: $\boxed{H(z) = 8 + \frac{8+7z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}}$

• $$8 + \frac{8+7z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} = 8 + \frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}} \Leftrightarrow$$

$$\Leftrightarrow \boxed{8+7z^{-1} = A\left(1+\frac{1}{2}z^{-1}\right) + B\left(1-\frac{1}{4}z^{-1}\right)}$$

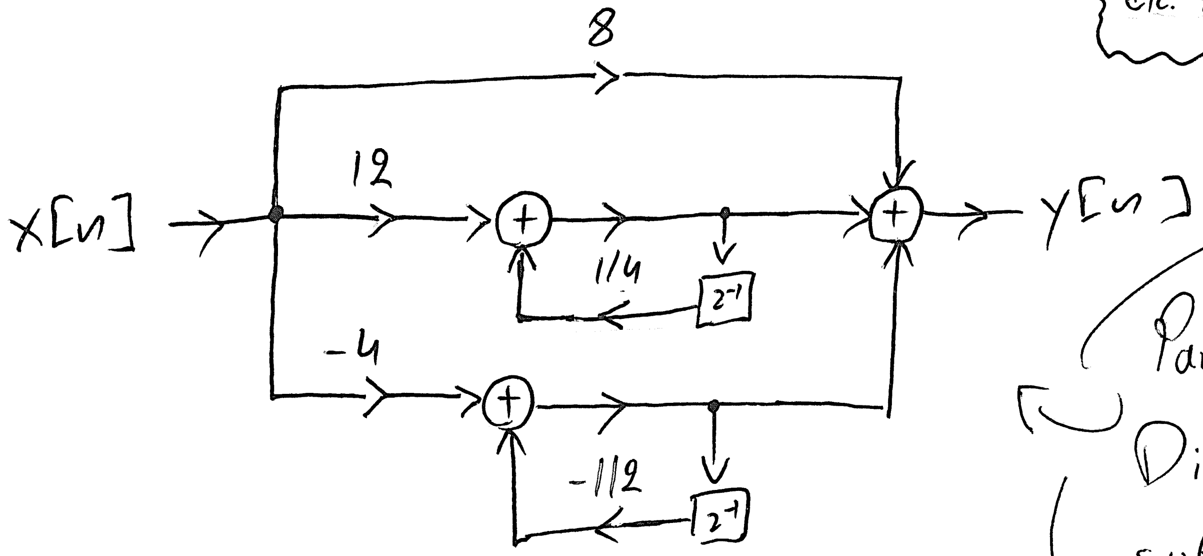
• $z^{-1}=-2$: $8+7(-2) = 0 + B\left(1-\frac{1}{4}(-2)\right) \Leftrightarrow$

$\Leftrightarrow 8-14 = B\left(1+\frac{1}{2}\right) \Leftrightarrow B = -6\cdot 2/3 \Leftrightarrow B=-4$

• $z^{-1}=4$: $8+7\cdot 4 = A\left(1+\frac{1}{2}4\right)+0 \Leftrightarrow 8+28 = A\cdot 3 \Leftrightarrow A=12$

So: $\boxed{H(z) = 8 + \frac{12}{1-\frac{1}{4}z^{-1}} - \frac{4}{1+\frac{1}{2}z^{-1}}}$

see 16.53 and 16.54 figures for Parallel Form, etc. for Direct Form I



Parallel Form I /w Direct Form I subsystems graph

• For Direct Form I we draw the numerator first (left-to-right).

Exercise 5

A 3-rd order FIR filter has the following transfer function:

$$G(z) = (6 - z^{-1} - 12z^{-2})(2 + 5z^{-1})$$

(a) Find all $H_i(z)$ that are FIR s.t. $|H_i(e^{j\omega})| = |G(e^{j\omega})|$.

• Let us factorize $G(z)$ first:

$6 - z^{-1} - 12z^{-2} = 0 \Leftrightarrow 6z^2 - z + 12 = 0$, $\Delta = 1 + 4 \cdot 6 \cdot 12 = 289 = 17^2$

$z_{1,2} = \frac{1 \pm 17}{12}$, $z_1 = \frac{18}{12} = \frac{3}{2}$, $z_2 = -\frac{16}{12} = \frac{4}{3}$

$\underline{6 - z^{-1} - 12z^{-2} = 6\left(1 - \frac{3}{2}z^{-1}\right)\left(1 - \frac{4}{3}z^{-1}\right)}$

Hence: $\boxed{G(z) = 30\left(1-\frac{3}{2}z^{-1}\right)\left(1-\frac{4}{3}z^{-1}\right)\left(1+\frac{5}{2}z^{-1}\right)}$

↳ all poles at $z=0$ ↳ zeros: $z=\frac{3}{2}$, $z=\frac{4}{3}$, $z=-\frac{5}{2}$

- We essentially want to change the locations of the zeros of $G(z)$, <u>without</u> altering its amplitude response $|G(e^{j\omega})|$.

! Changing a zero or a pole to its mutual conjugate does <u>not</u> alter the overall amplitude response of the system, iff it is done by the following way:

$$\boxed{1-az^{-1} \rightarrow z^{-1}-a^*, \quad a\in\mathbb{C}}$$

is zero at $z=a$ $\rightarrow$ is zero at $z=1/a^*$ $\Rightarrow$ <u>same</u> amplitude response whether a is a pole or zero.

- <u>Warning</u>: This change looks as though it achieves the same:

$a\in\mathbb{C}$, $1-az^{-1} \rightarrow 1-(1/a^*)z^{-1}$

is zero at $z=a$ — is zero at $z=1/a^*$ $\Rightarrow$ <u>not</u> the same amplitude response whether a is a pole or zero.

Proof with a zero at $z=a$, $a\in\mathbb{R}$ (as in the exercise):

- $H_1(z) = 1 - az^{-1}$, $H_2(z) = z^{-1} - a$, $a \in \mathbb{R}$.

- $H_1(e^{j\omega}) = H_1(z)\Big|_{z=e^{j\omega}} = 1 - ae^{-j\omega} = 1 - a\cos(\omega) - j\sin(\omega)\cdot a$

- $H_2(e^{j\omega}) = H_2(z)\Big|_{z=e^{j\omega}} = e^{-j\omega} - a = \cos(\omega) - a + j\sin(\omega)$

- $$\frac{|H_1(e^{j\omega})|}{|H_2(e^{j\omega})|} = \frac{\sqrt{(1-a\cos(\omega))^2 + (-a\sin(\omega))^2}}{\sqrt{(\cos(\omega)-a)^2 + (\sin(\omega))^2}} =$$

$$= \frac{1 - 2a\cos(\omega) + a^2\cos^2(\omega) + a^2\sin^2(\omega)}{\cos^2(\omega) - 2a\cos(\omega) + a^2 + \sin^2(\omega)} =$$

$$= \frac{1 - 2a\cos(\omega) + a^2(\cos^2(\omega) + \sin^2(\omega))}{\cos^2(\omega) + \sin^2(\omega) - 2a\cos(\omega) + a^2} =$$

$$= \frac{1 - 2a\cos(\omega) + a^2}{1 - 2\cos(\omega) + a^2} = 1 \Leftrightarrow \underline{|H_1(e^{j\omega})| = |H_2(e^{j\omega})|}$$

$e^{jx} = \cos(x) + j\sin(x)$ $|z| = \sqrt{(\mathrm{Re}\{z\})^2 + (\mathrm{Im}\{z\})^2}$, $z \in \mathbb{C}$

$\cos^2(x) + \sin^2(x) = 1$

• In this exercise, we have a total of $N = 3$ zeroes, and we can either choose to change it to its mutual conjugate or not, in order for $|G(e^{j\omega})|$ to remain the same. Hence, we can come up with $2^N = 2^3 = 8$ systems, such that:

$$|H_i(e^{j\omega})| = |G(e^{j\omega})| \quad , \quad 1 \leq i \leq 8.$$ These are:

• $H_1(z) = A(1-az^{-1})(1-bz^{-1})(1-cz^{-1})$,

• $H_2(z) = A(1-az^{-1})(1-bz^{-1})(z^{-1}-c^*)$,

• $H_3(z) = A(1-az^{-1})(z^{-1}-b^*)(1-cz^{-1})$,

• $H_4(z) = A(1-az^{-1})(z^{-1}-b^*)(z^{-1}-c^*)$,

• $H_5(z) = A(z^{-1}-a^*)(1-bz^{-1})(1-cz^{-1})$,

• $H_6(z) = A(z^{-1}-a^*)(1-bz^{-1})(z^{-1}-c^*)$,

• $H_7(z) = A(z^{-1}-a^*)(z^{-1}-b^*)(1-cz^{-1})$,

• $H_8(z) = A(z^{-1}-a^*)(z^{-1}-b^*)(z^{-1}-c^*)$, where:

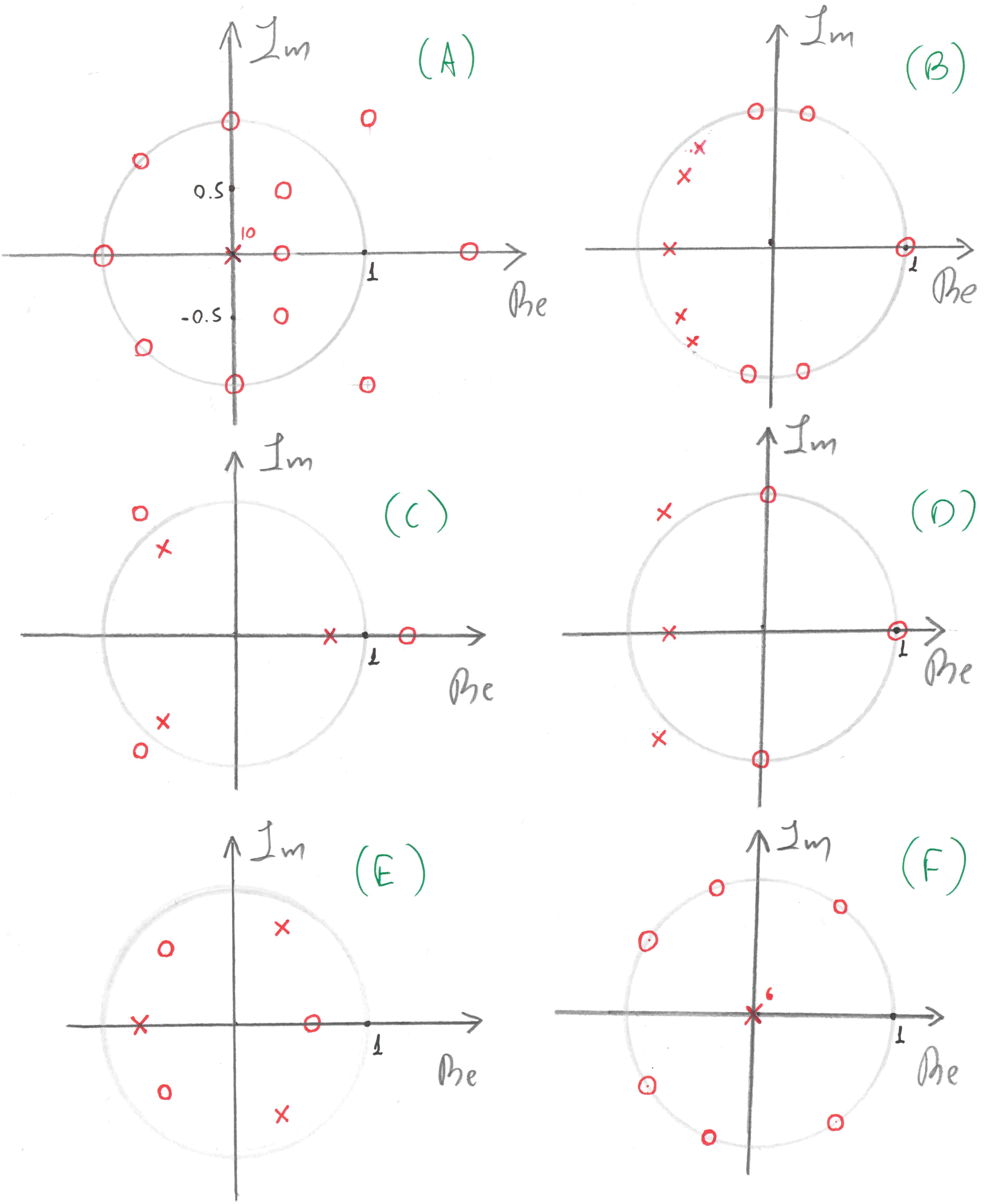
$A = 30$, $a = \frac{3}{2}$, $b = \frac{4}{3}$, $c = -\frac{5}{2}$, in the exercise.

(b) Which of these systems is minimum phase and which is maximum phase?

- Minimum phase $\Leftrightarrow$ stable and zeroes & poles in unit circle
  - $\hookrightarrow$ All poles here are at $z=0$, already in unit circle.
  - $\hookrightarrow$ For all zeroes to also be in the unit circle, we require for them to be at $z=2/3$, $z=3/4$, $z=-2/5$, i.e., all of them flipped compared to $G(z)$.
  - $\hookrightarrow$ Only system that meets the above criteria is $\underline{H_8(z)}$.
  - $\hookrightarrow$ For stability $\Leftrightarrow$ unit circle in ROC, which is met by the location of poles, i.e, the z transform converges for all values except $z=0$, so the unit circle is included.
- Maximum phase $\Leftrightarrow$ stable & causal & zeroes out of unit circle
  - $\hookrightarrow$ $z \neq 0$ for the ROC guarantees both causality & stability, i.e, unit circle in ROC & $|z| > a$, where $a=0$ here.
  - $\hookrightarrow$ For all zeroes to be outside the unit circle, we require for them to be at $z=3/2$, $z=4/3$, $z=-5/2$, identical locations with $G(z)$.
  - $\hookrightarrow$ Only system that is maximum phase, therefore, is $\underline{H_1(z)}$.

Exercise 6

Six different LTI systems have the following pole-zero plots:



Which of these systems:

(a) are IIR?

(b) are FIR?

(c) are stable (BIBO)?

(d) are minimum phase?

(e) are generalized linear phase?

(f) have $|H(e^{j\omega})| = a$, constant $\forall \omega \in [-\pi, \pi]$?

(g) have stable and causal inverse $H^{-1}(e^{j\omega})$?

(h) has the smallest (in length) $h[n]$ in duration?

(i) has spectral response with low-pass behavior?

(j) is minimum group delay?

—

(a) IIR $\Leftrightarrow$ exists pole at neither zero or infinity, so:

B, C, D, E systems are IIR.

(b) FIR $\Leftrightarrow$ all poles are at zero or infinity, so:

A, F systems are FIR.

(c) stability (BIBO) $\Leftrightarrow$ unit circle in ROC, so: all systems A, B, C, D, E, F can be stable, since none have a pole at the unit circle. For system D, the ROC will be $a < |z| < b$, where at a is a pole outside the unit circle, and at b the pole inside the unit circle. For all other systems, the poles are all inside the unit circle, hence $|z| > c$, will satisfy stability, where c is the location of the pole with the largest norm.

(d) Minimum phase $\Leftrightarrow$ stable and poles & zeroes in the unit circle, so: E is the only system that satisfies all criteria mentioned.

(e) Generalized linear phase $\Leftrightarrow$ 1) FIR (for this course), and we know that the systems are causal, so, thus far, systems A, F satisfy these conditions. 2) Complex zeroes with $|z_0| \neq 1$ occur in groups of four: $z_0, z_0^*, 1/z_0, 1/z_0^*$, met by A, F. 3) Complex zeroes on the unit circle occur as complex conjugate pairs: z_0, z_0^*, also met by A, F systems. 4) Real zeroes with $|z_0| \neq 1$ occur in mutual pairs: $z_0, 1/z_0$, met by A, F. 5) Real zeroes on the unit circle (i.e., $z_0 = \pm 1$) do not imply another one, so, A, F are generalized linear phase.

(f) $|H(e^{j\omega})| = a$, constant $\forall \omega \in [-\pi, \pi]$ $\Leftrightarrow$ all-pass $\Leftrightarrow$

$\Leftrightarrow$ each pole should be paired with a conjugate reciprocal zero, i.e., if $z = a$ is a pole then $z = 1/a^*$ is a zero, for all poles. Only system C satisfies this, i.e., it is the only all-pass system. A and F are FIR but no pole at infinity is allowed since we know that all systems are causal. B, D, E systems do not have the corresponding zero that is demanded of all-pass systems for each one of their poles.

(g) Stable & causal inverse $\Leftrightarrow$ flip zeroes and poles and check if ROC includes the unit circle and $|z| > a$ for the new (inverse) system. Minimum phase systems that are causal and stable automatically meet this condition, so E does have a stable and causal inverse. This is because there will always be a pole inside the unit circle with magnitude equal or less than all the other poles. Now, A, B, D, F systems cannot have a stable inverse because they have at least one zero on the unit circle, and C cannot have a stable and causal inverse because all its zeroes are outside the unit circle.

(h) Minimum duration impulse response: A and F are the only FIR systems, so all the others are automatically discarded. From (e), we showed that these systems are also generalized linear phase, so this makes our job much easier. We know that the length of these systems is $M+1$ samples in the time domain, where M is their number of zeroes. Hence, F has the smallest duration impulse response, since it has $M=6$ zeroes, in contrast to system A which has $M=10$.

(i) Low-pass behavior $\Leftrightarrow$ low frequencies boosted (poles), and/or high frequencies hindered (zeroes). Systems B, D are clearly high-pass due to the zero at $\omega=0$, and the poles at the high frequencies. System (C) is all-pass from (f), so it does not "favor" any frequency range over another (constant amplitude response), and, therefore, cannot be characterized as low-pass. System E could be better characterized as bandpass rather than low-pass. System F has zeroes scattered almost uniformly on the unit circle, but its zeroes are closer to the higher relatively to the lower frequencies, having, therefore, a low-pass behavior. Finally, A is "too chaotic" to tell, but it does not clearly exhibit neither low-pass or high-pass behavior.

(j) Minimum group delay $\Leftrightarrow$ Minimum phase $\Rightarrow$ same as (d), i.e., E.
(just another name for these systems)

END