Exercise 1

6º Φροντιστήριο
HY-370

Let S_1 be a causal and stable LTI system with impulse response $h_1[n]$ and frequency response $H_1(e^{jw})$, described by:

$$y[n] - y[n-1] + \frac{1}{4} y[n-2] = x[n], \quad (i).$$

I) If LTI system S_2 has frequency response $H_2(e^{jw}) = H_1(-e^{jw})$, is it highpass, lowpass, or bandpass?

- $\mathcal{Z}\{(i)\} \Rightarrow Y(z)\left(1 - z^{-1} + \frac{1}{4} z^{-2}\right) = X(z) \Rightarrow$

$$\Rightarrow H_1(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - z^{-1} + \frac{1}{4} z^{-2}} \Leftrightarrow$$

$$\Leftrightarrow \boxed{H_1(z) = \frac{1}{\left(1 - \frac{1}{2} z^{-1}\right)^2}}$$

- $H_2(e^{jw}) = H_1(-e^{jw}) \Rightarrow h_2[n] = -h_1[n] \Rightarrow \underline{H_2(z) = H_1(-z)}$

Hence: $\boxed{H_2(z) = \frac{1}{\left(1 + \frac{1}{2} z^{-1}\right)^2}}$ → Double pole at $z^{-1} = -2 \Leftrightarrow$

$$\Leftrightarrow z = -\frac{1}{2} \Leftrightarrow \boxed{z = \frac{1}{2} e^{j\pi}}$$

- This means that frequencies around $\omega = \pi$ are boosted, making $H_2(z)$ a highpass filter (poles make the z transform go towards infinity there).

II) Let S_3 be a causal LTI system with $H_3(e^{j\omega}) H_1(e^{j\omega}) = 1$. Is it minimum phase? Is it among the four types of linear phase systems that you know of?

- $H_3(e^{j\omega}) H_1(e^{j\omega}) = 1 \Rightarrow h_3[n] * h_1[n] = \delta[n] \Rightarrow$

$$\Rightarrow H_3(z) H_1(z) = 1 \Leftrightarrow \boxed{H_3(z) = \frac{1}{H_1(z)}}, \text{ hence:}$$

- $\boxed{H_3(z) = \left(1 - \frac{1}{2} z^{-1}\right)^2}$ → 2-nd rank zero at $\boxed{z = \frac{1}{2} e^{j0}}$

- Therefore, it hinders frequencies around $\omega = 0$, making it a highpass filter.
- Its zeros are in the unit circle, so it is also minimum phase.
- Its zeros are not in conjugate mutual pairs, making it not liner phase.

• We can also see the latter from the time domain:

$$Z^{-1}_{z}\{H_3(z)\} = h_3[n] = \delta[n] - \delta[n-1] + \frac{1}{4}\delta[n-2],$$

where no characteristic symmetry of FIR linear phase system is observed.

Exercise 2

• We have 3 causal and real LTI systems with transfer functions $H_1(z)$, $H_2(z)$, $H_3(z)$ respectively. Find as much information about them as you can, regarding:

(a) Their poles and zeroes.

(b) The duration of their impulse response.

I) It is given that: $H_1(z)$ has a pole at $z = 0.9\, e^{j\pi/3}$, and for input $x[n] = u[n]$, it holds that $\lim_{n \to +\infty} y[n] = 0$, where $y[n]$ is the output of the system.

(a) • The system is real, therefore its poles and zeroes come in conjugate pairs, or, its poles and zeroes are real.

• So, it must have another pole at $z^* = 0.9\, e^{-j\pi/3}$.

• Causal means that it has no poles at infinity.

- We know that: $x[n] = u[n] \Rightarrow X(z) = \frac{1}{1-z^{-1}}$, $|z| > 1$.

The output in this case at the z domain will have the form:

$Y(z) = H_1(z) X(z) = \frac{H_1(z)}{1-z^{-1}}$, and since we know that for this input, it holds that $\lim_{n \to +\infty} y[n] = 0$, this indicates that the pole at $z=1$ must be cancelled with a <u>zero at $z=1$</u>.

- Another way to think about it, is that if we perform partial fraction decomposition for $\frac{H_1(z)}{1-z^{-1}}$, then we will get a term $A \cdot \frac{1}{1-z^{-1}}$. In the time domain, this term translates to $\mathcal{Z}^{-1}\{A/1-z^{-1}\} = A u[n]$, but this term will not fade to 0 as $n \to +\infty$ (it will remain constant $= A$). Hence, to satisfy $\lim_{n \to +\infty} y[n] = 0$, this term must be cancelled, so $H_1(z)$ must have $1-z^{-1}$ in the numerator, i.e., a zero at $z=1$.

- Therefore, in total, we have:

$$H_1(z) = \frac{1-z^{-1}}{(1-0.9e^{j\pi/3}z^{-1})(1-0.9e^{-j\pi/3}z^{-1})} H_0(z),$$

where $H_0(z)$ keeps the rest of the information that we do not know about this system.

(b) Since there is at least one pole for $H_1(z)$ (that is not cancelled out), then its impulse response must be of <u>infinite duration</u>. A term of the form $\frac{A}{1-az^{-1}}$, $0<|a|<1$, for example, will transform into $Aa^n u[n]$, which is non-zero $\forall n \geq 0$, thus continuing indefinetely.

II) $H_2(z)$ has a zero at $z=0.8e^{j\pi/4}$, it is linear phase with group delay $\text{grd}\{H_2(e^{j\omega})\} = 2.5$, and $|H_2(e^{j0})|=0$.

(a) • Again, real means that zeroes come in mutual conjugate pairs $(z, z^*, \frac{1}{z}, \frac{1}{z^*})$, so it will have zeroes at $(0.8e^{j\pi/4}, 0.8e^{-j\pi/4}, 1.25e^{j\pi/4}, 1.25e^{-j\pi/4})$.

• Causal + linear phase means that it has to be one of the four types that we know from theory. We have:

$\text{grd}\{H_2(e^{j\omega})\} = 2.5 = \frac{5}{2}$, so:

$\measuredangle H_2(e^{j\omega}) = -\omega\frac{5}{2} + C$, where C is a constant.

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$ {Group delay = negative derivative of the phase.

- Thus, $M = 5$ in our case, which is an odd number. So, this is an either type II or type IV linear phase system. Therefore, all of its poles are located at the origin $z=0$.

- $|H_2(e^{j0})| = 0$ means that at $\omega = 0$, or $z = 1$ there is a zero. That makes it a type IV linear phase system.

- Since $M = 5$, there is a total of 5 zeroes for $H_2(z)$ and we have found all 5 of them. In total, we can say that:

$$H_2(z) = A(1 - z^{-1})(1 - z_1)(1 - z_1^*)(1 - 1/z)(1 - 1/z^*),$$

where $z_1 = 0.8e^{j\pi/4}$, $A \in \mathbb{R}$, almost fully describing this system.

(b) From theory, the duration of all type I, II, III, IV linear phase systems' $h[n]$ is $M+1$ samples. Hence, $h_2[n]$ has 6 delta functions, i.e., its duration is 6 samples long.

III) $H_3(z)$ has a pole at $z = 0.8e^{j\pi/4}$, and $|H_3(e^{j\omega})| = 1$, $\forall \omega$.

(a) $|H_3(e^{j\omega})| = 1$, $\forall\omega$ => this is an all-pass system.

Therefore, its zeroes and poles come in conjugate mutual pairs:

Pole at $z = 0.8e^{j\pi/4}$ => pole at $z^* = 0.8e^{-j\pi/4}$ =>

zeroes at $1/z = 1.25e^{j\pi/4}$, $1/z^* = 1.25e^{-j\pi/4}$.

Hence, we collected that:

$$H_3(z) = \frac{(z^{-1}-0.8e^{j\pi/4})(z^{-1}-0.8e^{-j\pi/4})}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})} H_{ap}(z),$$

where $H_{ap}(z)$ holds the rest of the information that we do not know about $H_3(z)$, and it is likewise also necessarily all-pass.

(b) Similarly, when we have a system with a pole that does not cancel out, then its impulse response is of infinite duration (IIR). Remember, a pole represents a term in the transfer function that causes a system's response to be dominated by an exponential decay or growth, depending on its position in the z-plane. Only a zero in the same position would nullify the pole's effect. Important: This is true, of course, for poles not at zero or infinity. If a system's poles are all located at zero or infinity, then it is FIR.

Exercise 3

Two LTI systems, $H_1(e^{j\omega})$, and $H_2(e^{j\omega})$, have generalized linear phase. Which of the following systems are also generalized linear phase?

(a) $G_1(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$

• The phase of this system is given by:

$$\measuredangle G_1(e^{j\omega}) = \tan^{-1}\left(\frac{\mathrm{Im}\{H_1(e^{j\omega}) + H_2(e^{j\omega})\}}{\mathrm{Re}\{H_1(e^{j\omega}) + H_2(e^{j\omega})\}}\right),$$

which is not always linear. We can give an example:

$h_1[n] = \delta[n] + 2\delta[n-1]$, $h_2[n] = 2\delta[n] - 2\delta[n-1]$.

It will be: $g_1[n] = h_1[n] + h_2[n] = 3\delta[n] - \delta[n-1]$, so:

$G_1(e^{j\omega}) = \mathcal{F}\{3\delta[n] - \delta[n-1]\} = 3 - e^{-j\omega} = 3 - \cos(\omega) + j\sin(\omega)$.

Hence: $\measuredangle G_1(e^{j\omega}) = \tan^{-1}\left(\frac{\sin(\omega)}{3-\cos(\omega)}\right)$, not a linear phase,

but, $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ are linear phase systems.

(b) $G_2(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})$

• The phase will be $\measuredangle G_2(e^{j\omega}) = \measuredangle H_1(e^{j\omega}) + \measuredangle H_2(e^{j\omega})$. Summing two lines always gives anoter line, so this system is always a generalized linear phase system.

(c) $G_3(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_1(e^{j\theta})H_2(e^{j(\omega-\theta)})\,d\theta.$

• A counter-example is an easy way to prove that this system is not necessarily generalized linear phase:

Remember that: $\mathcal{F}\{x[n]\,y[n]\} = X(e^{jw}) * Y(e^{jw}) =$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\,Y(e^{j(w-\theta)})\,d\theta, \text{ by definition.}$$

Therefore, for this system: $g_3[n] = h_1[n]\,h_2[n]$.

- As for the counter-example, we take two linear phase systems:

$h_1[n] = \delta[n] + \delta[n-1]$, $h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$.

So: $g_3[n] = (\delta[n] + \delta[n-1])(\delta[n] + 2\delta[n-1] + \delta[n-2])$

$$x[n-n_1]\,\delta[n-n_2] = x[n_2-n_1]\,\delta[n-n_2]$$

$$= \delta[0]\,\delta[n] + \delta[1]\cdot 2\delta[n-1] + \delta[2]\,\delta[n-2] +$$

$$+ \delta[-1]\,\delta[n] + \delta[0]\,2\delta[n-1] + \delta[1]\,\delta[n-2]$$

$$= \delta[n] + 2\delta[n-1] = g_3[n].$$

- Thus: $G_3(e^{jw}) = \mathcal{F}\{\delta[n] + 2\delta[n-1]\} = 1 + 2e^{-jw} = 1 + 2\cos(w) - 2j\sin(w)$,

giving a non-linear phase: $\measuredangle G_3(e^{jw}) = \tan^{-1}\left(\frac{2\sin(w)}{1+2\cos(w)}\right)$.

$\delta[n-n_0] \xrightarrow{\mathcal{F}} e^{-jwn_0} = \cos(w\cdot n_0) - j\sin(w\cdot n_0)$, from:

$e^{jx} = \cos(x) + j\sin(x)$.

Exercise 4 (from 2016 HY-370 final exam)

The Fibonacci sequence $(0,1,1,2,3,5,8,\ldots)$ can be modelled as the impulse response of the following system:

$$y[n]-y[n-1]-y[n-2]=x[n-1] \quad , (i)$$

Calculate this impulse response $h[n]$.

—

• $\mathcal{Z}\{(i)\} \Leftrightarrow Y(z)\left(1-z^{-1}-z^{-2}\right)=X(z)\,z^{-1} \Leftrightarrow$

$$\Leftrightarrow \boxed{H(z)=\frac{Y(z)}{X(z)}=\frac{z^{-1}}{1-z^{-1}-z^{-2}} \quad , (ii)} \; ,$$

with poles at $1-z^{-1}-z^{-2}=0 \Leftrightarrow z^2-z-1=0$

$\Delta=(-1)^2-4\cdot1\cdot(-1)=5$, $\boxed{z_{1,2}=\frac{1\pm\sqrt{5}}{2} \quad , (iii)}$

• Partial Fraction Decomposition on (ii):

$$\frac{z^{-1}}{1-z^{-1}-z^{-2}}=\frac{A}{1-z_1\cdot z^{-1}}+\frac{B}{1-z_2 z^{-1}} \Leftrightarrow z^{-1}=A\left(1-z_2z^{-1}\right)+B\left(1-z_1z^{-1}\right).$$

• For $z^{-1}=z_2^{-1}$: $\frac{1}{z_2}=B\left(1-\frac{z_1}{z_2}\right) \Leftrightarrow B=\frac{1}{z_2-z_1} \overset{(iii)}{\Rightarrow} \underline{B=-1/\sqrt{5}}$

• For $z^{-1}=z_1^{-1}$: $\frac{1}{z_1}=A\left(1-\frac{z_2}{z_1}\right) \Leftrightarrow A=\frac{1}{z_1-z_2} \overset{(iii)}{\Rightarrow} \underline{A=1/\sqrt{5}}$

So, in total:

$$\boxed{H(z) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{1+\sqrt{5}}{2}z^{-1}}\right) - \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{1-\sqrt{5}}{2}z^{-1}}\right), \quad (iv)}$$

- But, how can we do the inverse without knowing the ROC?
  - Observe that the Fibonacci sequence increases indefinetely, so this system is <u>not stable</u>. => unit circle not in ROC.
  - From (i), we can see that there is no dependence on future values of the input in order to get the output for this system. Hence, this system is <u>causal</u>. => $|z| > a$, $a \in \mathbb{R}$.
  - From the above, the ROC is $|z| > \frac{1+\sqrt{5}}{2} \simeq 1.618\cdots$, so now we can use the corresponding known pairs to get to the time domain:

$$Z^{-1}\{(iv)\} \Leftrightarrow \boxed{h[n] = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n u[n] - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n u[n].}$$

<u>Exercise 5</u> (from HY-370 2016 final exam)

The real part of the Fourier transform of a real and causal signal $x[n]$ is $X_R(e^{j\omega}) = 2 - 4\cos(3\omega)$. Find $X(e^{j0})$ and $X(e^{j\pi})$.

• From our theory (see table 13.4 page 721), we know that for a real signal $x[n]$, the real part of its Fourier transform corresponds to its even part, i.e., $x_e[n] \overset{F}{\longleftrightarrow} X_R(e^{j\omega})$.

This is called a Hermitian property for $X(e^{j\omega})$.

• So we can find this even part:

$$X_R(e^{j\omega}) = 2 - 4\cos(3\omega) = 2 - 2e^{j3\omega} - 2e^{-j3\omega} \quad F^{-1}$$

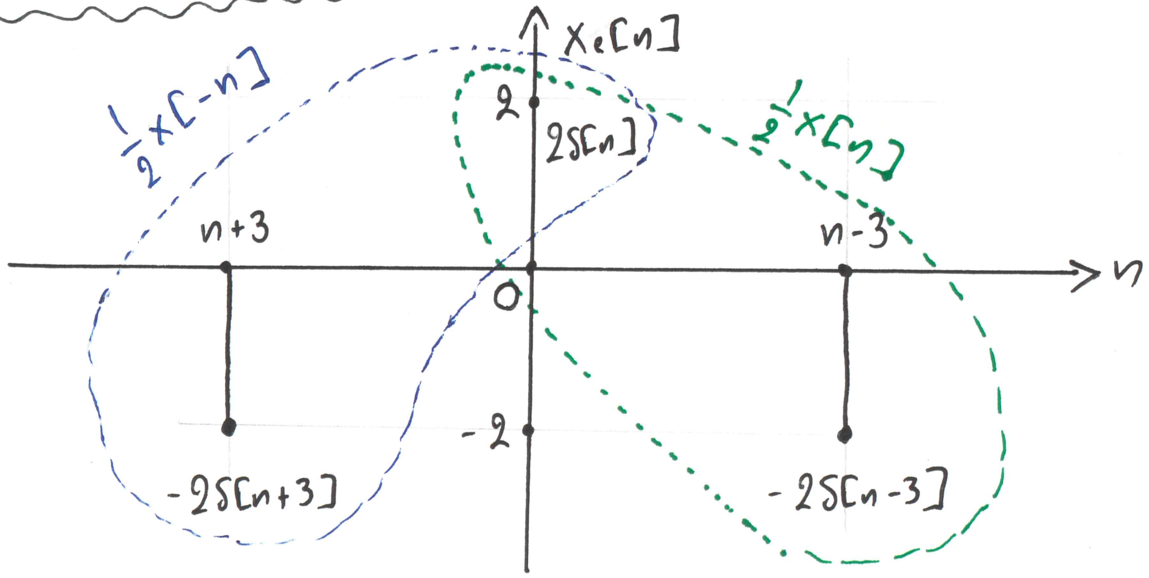
$$\boxed{x_e[n] = 2\delta[n] - 2\delta[n+3] - 2\delta[n-3].}$$

• Even part means that: $\boxed{x_e[n] = \frac{1}{2}x[n] + \frac{1}{2}x[-n]}$, or:

$$\boxed{x[n] = 2x_e[n] - x[-n].}$$

For the odd part, it would be $x_o[n] = \frac{1}{2}x[n] - \frac{1}{2}x[-n]$.

Notice that $x_e[n] + x_o[n] = \frac{1}{2}x[n] + \frac{1}{2}x[-n] + \frac{1}{2}x[n] - \frac{1}{2}x[-n] = x[n]$



- Hence, we can see (also from the plot) that our signal: $\boxed{x[n] = 2\delta[n] - 4\delta[n-3]}$, which is causal as we know from the exercise.

- We can also see from the plot that $x[-n] = -2\delta[n+3] + 2\delta[n]$.
As a sanity check, we can plug it in $x[n] = 2x_e[n] - x[-n]$ to see if we get the same $x[n]$ algebraically like so:

$$x[n] = 2x_e[n] - x[-n] =$$

$$= 2\left(2\delta[n] - 2\delta[n+3] - 2\delta[n-3]\right) - \left(-2\delta[n+3] + 2\delta[n]\right)$$

! system is causal, so it must be zero for all $n < 0$,
therefore, $\delta[n+3]$ is discarded:

$$= 4\delta[n] - 4\delta[n-3] - 2\delta[n] = \boxed{2\delta[n] - 4\delta[n-3] = x[n]}$$

- Now, we can evaluate what's been asked:

$$X(e^{j0}) = \left.\sum_{n=-\infty}^{+\infty} x[n]\, e^{j\omega n}\right|_{\omega=0} = \sum_{n=-\infty}^{+\infty} x[n] = 2 - 4 = -2.$$

$$X(e^{j\pi}) = \left.\sum_{n=-\infty}^{+\infty} x[n]\, e^{j\omega n}\right|_{\omega=\pi} = \sum_{n=-\infty}^{+\infty} x[n](-1)^n = 2(-1)^0 - 4(-1)^3 =$$

$$= 2 + 4 = 6.$$

END