Exercise 1

5º Φροντιστήριο ΗΥ-370

LTI with transfer function: $H(z) = \dfrac{1+z^{-1}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})(1+3z^{-1})}$

Find the impulse response of this system for each possible ROC, and draw the pole-zero diagram and comment on its stability and causality for each case:

$$H(z) = H(z)\cdot z^3/z^3 = \frac{z^3(1+z^{-1})}{z(1-z^{-1})\,z(1-\frac{1}{2}z^{-1})\,z(1+3z^{-1})} =$$

Zeros:

$z=0$ (double)

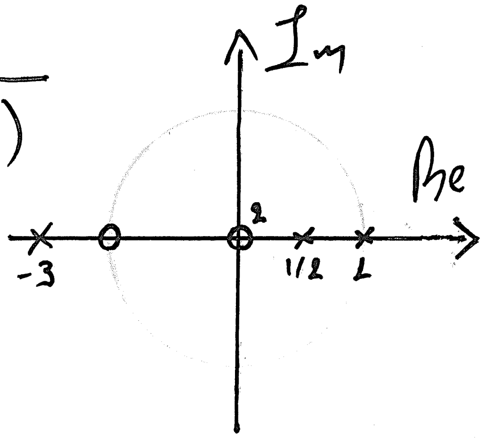
$z=-1$

Poles:

$z=1,\ z=\frac{1}{2},\ z=-3$

$$= \frac{z^3+z^2}{(z-1)(z-1/2)(z+3)}$$

$$= \frac{z^2(z+1)}{(z-1)(z-1/2)(z+3)}$$



↳ There are 4 possible ROCs:

a) $|z|<1/2$, b) $1/2<|z|<1$, c) $1<|z|<3$, d) $|z|>3$

(Remember, a ROC that is valid does not contain any poles.)

↳ To find the $h[n]$ for each case, we will use our known pairs and properties, so:

$$H(z) = \frac{1+z^{-1}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})(1+3z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}} + \frac{C}{1+3z^{-1}} \Leftrightarrow$$

$$1+z^{-1} = A\left(1-\tfrac{1}{2}z^{-1}\right)\left(1+3z^{-1}\right) + B\left(1-z^{-1}\right)\left(1+3z^{-1}\right) + C\left(1-z^{-1}\right)\left(1-\tfrac{1}{2}z^{-1}\right),$$ so:

- $z=1 : 1+1 = A\left(1-\frac{1}{2}\right)(1+3) + B\cdot 0 + C\cdot 0 \Leftrightarrow$

  $\Leftrightarrow 2 = A(1/2)(4) \Leftrightarrow A=1$

- $z=1/2 : 1+2 = A\cdot 0 + B(1-2)(1+3\cdot 2) + C\cdot 0 \Leftrightarrow$

  $\Leftrightarrow 3 = B(-1)(7) \Leftrightarrow B = -3/7$

- $z=-3 : 1-\frac{1}{3} = A\cdot 0 + B\cdot 0 + C\left(1+\frac{1}{3}\right)\left(1+\frac{1}{2}\cdot\frac{1}{3}\right) \Leftrightarrow$

  $\Leftrightarrow \frac{2}{3} = C\left(\frac{4}{3}\right)\left(\frac{7}{6}\right) \Leftrightarrow C = 3/7$, hence:

$$H(z) = \frac{1}{1-z^{-1}} - \frac{3}{7}\cdot\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{3}{7}\cdot\frac{1}{1+3z^{-1}}$$

$\hookrightarrow$ Now, depending on the ROC case, we choose the correct pairs for the inverse transform:

For case a), it will be:

$$h_a[n] = -u[-n-1] + \frac{2}{7}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{2}{7}(-3)^n u[-n-1]$$

Case b):

$$h_b[n] = -u[-n-1] - \frac{2}{7}\left(\frac{1}{2}\right)^n u[n] - \frac{2}{7}(-3)^n u[-n-1]$$

Case c):

$$h_c[n] = u[n] - \frac{2}{7}\left(\frac{1}{2}\right)^n u[n] - \frac{2}{7}(-3)^n u[-n-1]$$

Case d):

$$h_d[n] = u[n] - \frac{2}{7}\left(\frac{1}{2}\right)^n u[n] + \frac{3}{7}(-3)^n u[n]$$

- Stability (BIBO) $\Leftrightarrow$ Unit circle in ROC:

  None of our 4 ROCs contain the unit circle, hence, this system cannot be BIBO stable.*

- Causality $\Leftrightarrow$ ROC: $|z| > a$

  Only case d) ROC satisfies this criterion.

* Notice that we have a pole at $z=1$, so no possible ROC exists to make the system stable.

Exercise 2

If the following transfer functions belong to stable LTI systems, then determine in each case whether or not the system is causal without computing the inverse Z-transform:

a) $H_a(z) = \dfrac{1 - \frac{4}{3}z^{-1} + \frac{1}{2}z^{-2}}{z^{-1}\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} = H_a(z) \cdot \dfrac{z^3}{z^3} =$

roots:

$$= \frac{z^3 - \frac{4}{3}z^2 + \frac{1}{2}z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)} = \frac{z\left(z^2 - \frac{4}{3}z + \frac{1}{2}\right)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$$

← 1 real, 2 complex

← 2 real, other?

↳ Other root for the denominator is at infinity.

↳ So, this system has a pole at infinity.

↳ So, it cannot have a ROC like $|z| > a$ for it to be causal, because valid ROCs do not contain poles.

↳ Hence, this system can never be causal.

b) $H_b(z) = \dfrac{z - \frac{1}{2}}{z^2 + \frac{1}{2}z - \frac{3}{16}} = \dfrac{z - \frac{1}{2}}{\left(z + \frac{3}{4}\right)\left(z - \frac{1}{4}\right)}$, poles: $z = 1/4$, $z = -3/4$

↳ With ROC $|z| > 3/4$ the system is both stable, because it contains the unit circle, and causal ($|z| > a$).
($|z| > 1/4$ is invalid because we have a pole at $z = -3/4$)

c) $$H_c(z) = \frac{z+1}{z+\frac{4}{3}-\frac{1}{2}z^{-2}-\frac{2}{3}z^{-3}} = \frac{z+1}{z+\frac{4}{3}-z^{-2}\left(\frac{1}{2}+\frac{2}{3}z^{-1}\right)} =$$

$$= \frac{2z(z+1)}{2z\left(z+\frac{4}{3}\right)-z^{-2}\left(z+\frac{4}{3}\right)} = \frac{2z(z+1)}{\left(z+\frac{4}{3}\right)\left(2z-z^{-2}\right)} =$$

$$= \frac{2z^3(z+1)}{\left(z+\frac{4}{3}\right)\left(2z^3-1\right)}$$, poles: $z=-4/3$, $z=\sqrt[3]{1/2}\simeq 0.793$

↳ For causality, we would require $|z|>4/3$, but, this ROC does not include the unit circle, so it is not stable.

↳ $|z|>\sqrt[3]{1/2}$ is invalid because we have a pole at $z=-4/3$.

↳ No ROC exists, therefore, for this system to be both causal and stable.

Exercise 3

• We have a linear-phase discrete-time LTI system with frequency response $H(e^{j\omega})$ and real impulse response $h[n]$. Its group delay function is defined as:

$$\tau(\omega) = -\frac{d}{d\omega} \measuredangle H(e^{j\omega}),$$

where $\measuredangle H(e^{j\omega})$ has no discontinuities. Also, suppose that:

$$|H(e^{j\pi/2})| = 2, \quad \text{and} \quad \tau(\pi/2) = 2.$$

Determine the output of the system for inputs:

(a) $\cos\left(\frac{\pi}{2}n\right) = \left(e^{j\pi n/2} + e^{-j\pi n/2}\right)/2$

↳ $\tau(\pi/2) = 2$ means that a complex exponential with frequency $\pi/2$ will be delayed by 2 samples.

↳ What about a complex exponential with frequency $-\pi/2$?

! When $h[n]$ is real, the group delay is an even function, so we can also deduce that $\tau(-\pi/2) = 2$ here.

↳ Hence, the output is given by delaying the two exponentials of our input by 2 samples, and multiplying by $|H(e^{j\pi/2})| = 2$:

$$y[n] = 2 \cdot \left(e^{j\pi(n-2)/2} + e^{-j\pi(n-2)/2}\right)/2 = 2\cos\left(\frac{\pi(n-2)}{2}\right)$$

$$= 2\cos\left(\frac{\pi n}{2} - \pi\right)$$

Warning: Here we cannot apply this to get the output:

$y[n] = |H(e^{j\pi/2})| \cdot \cos\left(\frac{\pi n}{2} + \measuredangle H(e^{j\pi/2})\right)$, because we do not know $\measuredangle H(e^{j\pi/2})$, only the negative derivative of it at $\pi/2$.

Counter example: If $\measuredangle H(e^{j\omega}) = -2\omega + 0.1$, then:

$\tau(\omega) = -\frac{d}{d\omega} \measuredangle H(e^{j\omega}) = 2$, but $\measuredangle H(e^{j\pi/2}) = -\pi + 0.1$.

Essentially, any constant will get lost at the derivation, so we cannot know if there was any. There is an infinite number of possible $\measuredangle H(e^{j\omega})$ that would delay the input by 2 samples, and we cannot claim that simply $\measuredangle H(e^{j\pi/2}) = -\pi$.

(b) $x[n] = \sin\left(\frac{7\pi n}{2} + \frac{\pi}{4}\right)$

- We only know what happens at frequencies $\pm\pi/2$ for complex exponentials, hence:

$$\sin\left(\frac{7\pi n}{2} + \frac{\pi}{4}\right) = \sin\left(\left(4 - \frac{1}{2}\right)\pi n + \frac{\pi}{4}\right) = \sin\left(4\pi n - \frac{\pi n}{2} + \frac{\pi}{4}\right) =$$

$$= -\sin\left(\frac{\pi n}{2} + \frac{\pi}{2}\right), \quad \sin(4\pi n + a) = \sin(a), \ n \in \mathbb{Z}.$$

↳ Hence, similarly, we delay it by 2 samples and multiply by the amplitude response to get the output:

$$y[n] = 2\sin\left(\frac{7\pi(n-2)}{2} + \frac{\pi}{4}\right)$$

$$= 2\sin\left(\frac{7\pi n}{2} - \frac{14\pi}{2} + \frac{\pi}{4}\right)$$

$$= 2\sin\left(\frac{7\pi n}{2} - 7\pi + \frac{\pi}{4}\right)$$

$$= 2\sin\left(\frac{7\pi n}{2} - \pi + \frac{\pi}{4}\right)$$

$$= 2\sin\left(\frac{7\pi n}{2} - \frac{3\pi}{4}\right).$$

Exercise 4

Causal LTI with frequency response:

$$H(e^{j\omega}) = e^{-j\omega} \frac{1 - \frac{1}{2} e^{j\omega}}{1 - \frac{1}{2} e^{-j\omega}}$$

(a) Show that $|H(e^{j\omega})| = 1$, $\forall \omega$:

$$|H(e^{j\omega})| = |e^{-j\omega}| \frac{|1 - \frac{1}{2} e^{j\omega}|}{|1 - \frac{1}{2} e^{-j\omega}|} = 1 \cdot 1 = 1,$$ conjugates

have the same norm, i.e., $|a + bj| = |a - bj|$, $\square$.

(b) Show that $\measuredangle H(e^{j\omega}) = -\omega - 2\tan^{-1}\left(\frac{\frac{1}{2}\sin(\omega)}{1 - \frac{1}{2}\cos(\omega)}\right)$:

$$\measuredangle H(e^{j\omega}) = \measuredangle e^{-j\omega} + \measuredangle\left(1 - \frac{1}{2} e^{j\omega}\right) - \measuredangle\left(1 - \frac{1}{2} e^{-j\omega}\right) =$$

$\measuredangle(a + jb) = \arctan(b/a) = \tan^{-1}(b/a)$

$$= -\omega + \measuredangle\left(1 - \frac{1}{2}\cos(\omega) - \frac{1}{2} j \sin(\omega)\right) + \measuredangle\left(1 - \frac{1}{2}\cos(\omega) + \frac{1}{2} j \sin(\omega)\right)$$

$$= -\omega + \tan^{-1}\left(\frac{-\frac{1}{2}\sin(\omega)}{1 - \frac{1}{2}\cos(\omega)}\right) - \tan^{-1}\left(\frac{\frac{1}{2}\sin(\omega)}{1 - \frac{1}{2}\cos(\omega)}\right)$$

$$= -\omega - 2\tan^{-1}\left(\frac{\frac{1}{2}\sin(\omega)}{1 - \frac{1}{2}\cos(\omega)}\right) \quad \square.$$

$\tan^{-1}(-x) = -\tan^{-1}(x)$

c) Show that the group delay of the system is:

$$T(\omega) = \frac{\frac{3}{4}}{\frac{5}{4} - \cos(\omega)}$$

• By definition: $\tau(\omega) = -\frac{d}{d\omega} \measuredangle H(e^{j\omega}) =$

$$= -\frac{d}{d\omega}\left(-\omega - 2\tan^{-1}\left(\frac{\frac{1}{2}\sin(\omega)}{1-\frac{1}{2}\cos(\omega)}\right)\right) =$$

$$= 1 + 2\frac{d}{d\omega}\tan^{-1}\left(\frac{\frac{1}{2}\sin(\omega)}{1-\frac{1}{2}\cos(\omega)}\right) =$$

$$\left\{ \frac{d}{dx}\tan^{-1}(f(x)) = \frac{1}{1+f^2(x)} \cdot f'(x) \right.$$

$$= 1 + 2\underbrace{\frac{1}{1+\left(\frac{\frac{1}{2}\sin(\omega)}{1-\frac{1}{2}\cos(\omega)}\right)^2}}_{(1)} \cdot \underbrace{\frac{d}{d\omega}\left(\frac{\frac{1}{2}\sin(\omega)}{1-\frac{1}{2}\cos(\omega)}\right)}_{(2)}, \quad (i).$$

$$(1):\ 1+\frac{\left(\frac{1}{2}\sin(\omega)\right)^2}{\left(1-\frac{1}{2}\cos(\omega)\right)^2}=\frac{\left(1-\frac{1}{2}\cos(\omega)\right)^2+\left(\frac{1}{2}\sin(\omega)\right)^2}{\left(1-\frac{1}{2}\cos(\omega)\right)^2}=$$

$$\sin^2(x)+\cos^2(x)=1$$

$$=\frac{1-\cos(\omega)+\frac{1}{4}\cos^2(\omega)+\frac{1}{4}\sin^2(\omega)}{\left(1-\frac{1}{2}\cos(\omega)\right)^2}=$$

$$=\frac{1-\cos(\omega)+\frac{1}{4}}{\left(1-\frac{1}{2}\cos(\omega)\right)^2}=\frac{\frac{5}{4}-\cos(\omega)}{\left(1-\frac{1}{2}\cos(\omega)\right)^2},\quad (ii).$$

$$\left(\frac{f(x)}{g(x)}\right)'=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$$

$$(2):\ \frac{d}{d\omega}\left(\frac{\frac{1}{2}\sin(\omega)}{1-\frac{1}{2}\cos(\omega)}\right)=$$

$$=\frac{\left(1-\frac{1}{2}\cos(\omega)\right)\left(\frac{1}{2}\cos(\omega)\right)-\left(\frac{1}{2}\sin(\omega)\right)\left(\frac{1}{2}\sin(\omega)\right)}{\left(1-\frac{1}{2}\cos(\omega)\right)^2}=$$

$$=\frac{\frac{1}{2}\cos(\omega)-\frac{1}{4}\cos^2(\omega)-\frac{1}{4}\sin^2(\omega)}{\left(1-\frac{1}{2}\cos(\omega)\right)^2}=\frac{\frac{1}{2}\cos(\omega)-\frac{1}{4}}{\left(1-\frac{1}{2}\cos(\omega)\right)},\quad (iii).$$

$$(i) \overset{(ii)}{\underset{(iii)}{\Longrightarrow}} T(\omega) = 1 + 2\left(\frac{\left(1 - \frac{1}{2}\cos(\omega)\right)^2}{\frac{5}{4} - \cos(\omega)}\right)\left(\frac{\frac{1}{2}\cos(\omega) - \frac{1}{4}}{\left(1 - \frac{1}{2}\cos(\omega)\right)^2}\right) =$$

$$= 1 + \frac{\cos(\omega) - \frac{1}{2}}{\frac{5}{4} - \cos(\omega)} = \frac{\frac{5}{4} - \cos(\omega) + \cos(\omega) - \frac{1}{2}}{\frac{5}{4} - \cos(\omega)} =$$

$$= \frac{\frac{3}{4}}{\frac{5}{4} - \cos(\omega)} \quad \square.$$

d) What is the output of the filter when the input is $x[n] = \cos(\pi n/3)$?

↳ From c) : $T\left(\pm\frac{\pi}{3}\right) = \dfrac{\frac{3}{4}}{\frac{5}{4} - \cos\left(\pm\frac{\pi}{3}\right)} = \dfrac{\frac{3}{4}}{\frac{5}{4} - \frac{1}{2}} = \dfrac{\frac{3}{4}}{\frac{3}{4}} = 1.$

↳ That means that a complex exponential with frequency $\pm\pi/3$ will be delayed by 1 sample.

↳ Here these are : $\cos(\pi n/3) = \left(e^{j\pi n/3} + e^{-j\pi n/3}\right)/2$

↳ Hence the output : $y[n] = \left(e^{j\pi(n-1)/3} + e^{-j\pi(n-1)/3}\right)/2$

$$= \cos\left(\frac{(n-1)\pi}{3}\right)$$

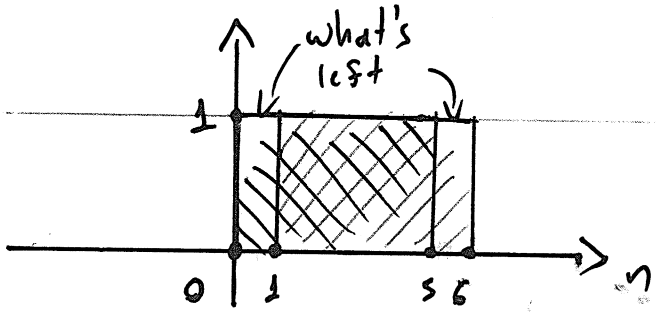
Exercise 5

$$x[n] = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases}, \quad g[n] = x[n] - x[n-1]$$

(a) Find $g[n]$ and its Z-transform:

$$g[n] = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases} - \begin{cases} 1, & 0 \le n-1 \le 5 \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases} - \begin{cases} 1, & 1 \le n \le 6 \\ 0, & \text{elsewhere} \end{cases}$$



$$= \begin{cases} 1, & n = 0 \\ -1, & n = 6 \\ 0, & \text{elsewhere} \end{cases} = \delta[n] - \delta[n-6]. \text{ So:}$$

$$Z\{g[n]\} = Z\{\delta[n]\} - Z\{\delta[n-6]\} = 1 - z^{-6}, \quad |z| > 0.$$

(6) If: $x[n] = \sum_{k=-\infty}^{n} g[n]$, find $Z\{x[n]\}$.

↳ property: $\sum_{k=-\infty}^{n} y[k] \xrightarrow{Z} \frac{1}{1-z^{-1}} Y(z)$, $R \supseteq \{R_x \cap \{|z|>1\}$

$$X(z) = \frac{1}{1-z^{-1}} \cdot G(z) = \frac{1-z^{-6}}{1-z^{-1}} = \frac{z^6-1}{z^6-z^5} = \frac{z(z+1)(z-1)(z^4+z^2+1)}{z^5(z-1)}$$

$$= \frac{(z+1)(z^4+z^2+1)}{z^4}$$, all poles at $z=0$, and

$x[n]$ is of finite duration, hence, $|z|>0$.

END