3º Φροντιστήριο
HY-370

Άσκηση 1) Using only properties & pairs of the Fourier Transform, show that:

$$\mathcal{F}\left\{\left(\frac{1}{4}\right)^{|n|}\right\} = \frac{\frac{15}{16}}{\frac{17}{16} - \frac{1}{2}\cos(\omega)}$$

↳ Express $(1/4)^{|n|}$ in a way where pairs appear:

$$\left(\frac{1}{4}\right)^{|n|} = \begin{cases} (1/4)^n, & n>0 \\ (1/4)^{-n}, & n<0 \\ 1, & n=0 \end{cases} = \begin{cases} (1/4)^n u[n-1] \\ (1/4)^{-n} u[-n+1] \\ \delta[n] \end{cases}$$

↳ Or we can include $n=0$ in the first two cases and then subtract the final case:

$$\left(\frac{1}{4}\right)^{|n|} = \left(\frac{1}{4}\right)^{n} u[n] + \left(\frac{1}{4}\right)^{-n} u[n] - \delta[n]$$, and from pairs:

$$\mathcal{F}\left\{\left(\frac{1}{4}\right)^{|n|}\right\} = \frac{1}{1-\frac{1}{4}e^{-j\omega}} + \frac{1}{1-\frac{1}{4}e^{j\omega}} - 1 =$$

$$= \frac{1-\frac{1}{4}e^{j\omega} + 1 - \frac{1}{4}e^{-j\omega}}{\left(1-\frac{1}{4}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{j\omega}\right)} - 1 =$$

$$= \frac{2 - \frac{1}{2}\cos(\omega)}{\left|1 - \frac{1}{4}e^{-j\omega}\right|^2} - 1 =$$

$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$

$z \cdot \bar{z} = |z|^2, \quad z \in \mathbb{C}$

Reminders

$|z|^2 = \mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2,$

$e^{jx} = \cos(x) + j\sin(x) \Leftrightarrow e^{-jx} = \cos(x) - j\sin(x)$

$\mathrm{Re}\{e^{jx}\} = \mathrm{Re}\{e^{-jx}\} = \cos(x)$

$\mathrm{Im}\{e^{jx}\} = -\mathrm{Im}\{e^{-jx}\} = \sin(x)$, so given these:

$$= \frac{2 - \frac{1}{2}\cos(\omega)}{\left(1 - \frac{1}{4}\cos(\omega)\right)^2 + \left(\frac{1}{4}\sin(\omega)\right)^2} - 1 =$$

$$= \frac{2 - \frac{1}{2}\cos(\omega)}{1 - \frac{1}{2}\cos(\omega) + \frac{1}{16}\cos^2(\omega) + \frac{1}{16}\sin^2(\omega)} - 1 =$$

$$= \frac{2 - \frac{1}{2}\cos(\omega)}{1 - \frac{1}{2}\cos(\omega) + \frac{1}{16}\left(\cos^2(\omega) + \sin^2(\omega)\right)} - 1 =$$

$\left\{ \cos^2(x) + \sin^2(x) = 1 \right.$

$$= \frac{2 - \frac{1}{2}\cos(\omega)}{1 + \frac{1}{16} - \frac{1}{2}\cos(\omega)} - 1 =$$

$$= \frac{2 - \frac{1}{2}\cos(\omega)}{\frac{17}{16} - \frac{1}{2}\cos(\omega)} - \frac{\frac{17}{16} - \frac{1}{2}\cos(\omega)}{\frac{17}{16} - \frac{1}{2}\cos(\omega)} =$$

$$= \frac{2 - \frac{17}{16}}{\frac{17}{16} - \frac{1}{2}\cos(\omega)} = \frac{\frac{15}{16}}{\frac{17}{16} - \frac{1}{2}\cos(\omega)} \quad \square.$$

Άσκηση 2) LTI system with input $x[n] = -3^n \cdot u[-n-1]$, and impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$. Find the system's output $y[n]$, by only using Fourier properties and pairs:

$$F\{-3^n u[-n-1]\} = \frac{1}{1-3e^{j\omega}}$$

$$F\left\{\left(\frac{1}{2}\right)^n u[n]\right\} = \frac{1}{1-\frac{1}{2}e^{j\omega}}$$

} using pairs that are known

Hence, knowing $Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega}) =$

$$= \frac{1}{1-3e^{j\omega}} \cdot \frac{1}{1-\frac{1}{2}e^{-j\omega}} = \frac{A}{1-3e^{-j\omega}} + \frac{B}{1-\frac{1}{2}e^{-j\omega}}$$

Partial Fraction Decomposition

$$\Leftrightarrow 1 = A\left(1-\frac{1}{2}e^{-j\omega}\right) + B\left(1-3e^{-j\omega}\right)$$

$e^{-j\omega} = 2 : \ 1 = A \cdot 0 + B(1-3\cdot 2) \Leftrightarrow \underline{B = -1/5}$

$e^{-j\omega} = 1/3 : \ 1 = A(1-1/6) + B \cdot 0 \Leftrightarrow \underline{A = 6/5}$

$$= \frac{6/5}{1-3e^{j\omega}} - \frac{1/5}{1-\frac{1}{2}e^{-j\omega}} = Y(e^{j\omega})$$

using known pairs $\mathcal{F}^{-1}$

$$y[n] = -\frac{6}{5}3^n u[-n-1] - \frac{1}{5}\left(\frac{1}{2}\right)^n u[n]$$

Ασκηση 3) Input to LTI system is given:

$$x[n] = 4\cos\left(\frac{\pi n}{6} + \frac{\pi}{8}\right) - \sin\left(\frac{\pi n}{4} - \frac{\pi}{2}\right)$$

(a') Find the output of the system $y[n]$, if its impulse response is: $h[n] = \dfrac{\sin\left(\frac{(n-2)\pi}{2}\right)}{(n-2)\pi}$

We know: $\mathcal{F}\left\{\dfrac{\sin\left(\pi \frac{n}{2}\right)}{\pi n}\right\} = \begin{cases} 1, & |\omega| \leq \pi/2 \\ 0, & \text{elsewhere} \end{cases}$, so:

$$H(e^{j\omega}) = \mathcal{F}\{h[n]\} = \mathcal{F}\left\{\frac{\sin\left(\frac{(n-2)\pi}{2}\right)}{(n-2)\pi}\right\} = \begin{cases} e^{-j2\omega}, & |\omega| \leq \pi/2 \\ 0, & \text{elsewhere} \end{cases},$$

from the time displacement property. We observe that

- $H(e^{j\omega})$ is a low-pass filter that cuts off frequencies not in $[-\pi/2, \pi/2]$, so both of our sinusoids are unaffected by this filter.
- We know the output form in case of sinusoid inputs for LTI systems (see previous tutorial for the formula), so we can compute the answer like so:

$$y[n] = \left|H(e^{j\pi/6})\right| 4\cos\left(\frac{\pi n}{6} + \frac{\pi}{8} + \measuredangle H(e^{j\pi/6})\right) -$$

$$\left|H(e^{j\pi/4})\right| \sin\left(\frac{\pi n}{4} - \frac{\pi}{2} + \measuredangle H(e^{j\pi/4})\right)$$

$$\left|H(e^{j\omega})\right| = \begin{cases} \left|e^{-j2\omega}\right|, & |\omega| \leq \pi/2 \\ |0|, & \text{elsewhere} \end{cases} = \begin{cases} 1, & |\omega| \leq \pi/2 \\ 0, & \text{elsewhere} \end{cases}$$

$$\measuredangle H(e^{j\omega}) = \begin{cases} \measuredangle e^{-j2\omega}, & |\omega| \leq \pi/2 \\ \text{undefined}, & \text{elsewhere} \end{cases} = -2\omega, \quad |\omega| \leq \pi/2$$

$$= 1 \cdot 4 \cdot \cos\left(\frac{\pi n}{6} + \frac{\pi}{8} - 2 \cdot \frac{\pi}{6}\right) - 1 \cdot \sin\left(\frac{\pi n}{4} - \frac{\pi}{2} - 2 \cdot \frac{\pi}{4}\right)$$

$$= 4\cos\left(\frac{\pi n}{6} + \frac{\pi}{8} - \frac{\pi}{3}\right) - \sin\left(\frac{\pi n}{4} - \frac{\pi}{2} - \frac{\pi}{2}\right)$$

$$= 4\cos\left(\frac{\pi n}{6} - \frac{5\pi}{24}\right) + \sin\left(\frac{\pi n}{4}\right).$$

$\sin(x - \pi)$
$= -\sin(x)$

(B') Compute the sum: $\sum_{n=-\infty}^{+\infty} \left| \frac{\sin(0.25\pi n)}{3\pi n} \right|^2$.

$\frac{1}{4} = 0.25$

↳ From Parseval's theorem:

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

↳ In our case:

$$x[n] = \frac{1}{3} \frac{\sin\left(\frac{\pi n}{4}\right)}{\pi n} \xrightarrow{F} X(e^{j\omega}) = \begin{cases} 1/3, & |\omega| \leq \pi/4 \\ 0, & \text{elsewhere} \end{cases}$$

↳ Hence:

$$\sum_{n=-\infty}^{+\infty} \left| \frac{\sin(0.25\pi n)}{3\pi n} \right|^2 = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} (1/3)^2 d\omega =$$

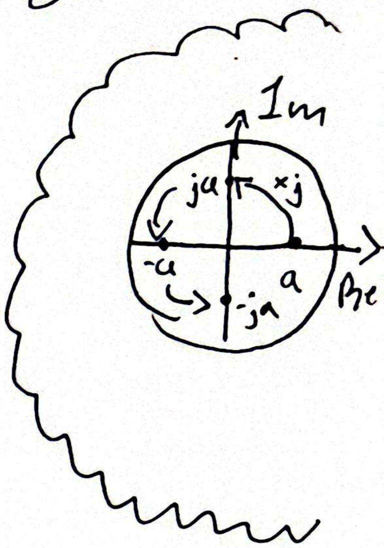
$$= \frac{1}{2\pi} \cdot \frac{1}{9} \left[\omega\right]_{-\pi/4}^{\pi/4} =$$

$$= \frac{1}{18\pi} \left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right) \right) =$$

$$= \frac{1}{18\pi} \cdot \frac{\pi}{2} = 1/36.$$

Άσκηση 4) A phase shifter by 90° can be expressed as an LTI system with the following frequency response:

$$H(e^{j\omega}) = \begin{cases} -j, & 0 < \omega < \pi \\ j, & -\pi < \omega < 0 \end{cases}$$



(a') Show that $\mathcal{F}^{-1}\{H(e^{j\omega})\} = \begin{cases} \frac{2}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

↳ Definition of inverse discrete fourier transform:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} j \cdot e^{j\omega n}\, d\omega - \frac{1}{2\pi}\int_{0}^{\pi} j \cdot e^{j\omega n}\, d\omega =$$

$$= \frac{j}{2\pi}\left(\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{0} - \left[\frac{e^{j\omega n}}{jn}\right]_{0}^{\pi}\right) =$$

$$= \frac{j}{2\pi}\left(\frac{1}{jn} - \frac{e^{-j\pi n}}{jn} - \left(\frac{e^{j\pi n}}{jn} - \frac{1}{jn}\right)\right)$$

$$= \frac{1}{2\pi n}\left(1 - e^{-j\pi n} - e^{j\pi n} + 1\right) = \quad , (i) \qquad \left\{ e^{\pm j\pi} = -1 \right.$$

$$= \frac{1}{2\pi n}\left(2 - 2(-1)^n\right) =$$

$$= \frac{1}{\pi n}\left(1 - (-1)^n\right) = \begin{cases} \frac{2}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \quad \square .$$

(B') Show that h[n] can be expressed as:

$$h[n] = \begin{cases} \frac{2}{\pi} \sin^2(\pi n/2), & n \neq 0 \\ 0, & n = 0 . \end{cases}$$

From (i), and $e^{j\pi n} + e^{-j\pi n} = 2\cos(\pi n)$:

$$h[n] = \frac{1}{2\pi n}\left(2 - 2\cos(\pi n)\right) =$$

$$= \frac{1}{\pi n}\left(1 - \cos(\pi n)\right) =$$

$\sin^2(x) = \frac{1-\cos(2x)}{2}$, power-reduction trigonometric formula

$$= \begin{cases} \frac{2}{\pi}\sin^2(\pi n/2) & , n \neq 0 \\ 0 & , n = 0 \end{cases}$$

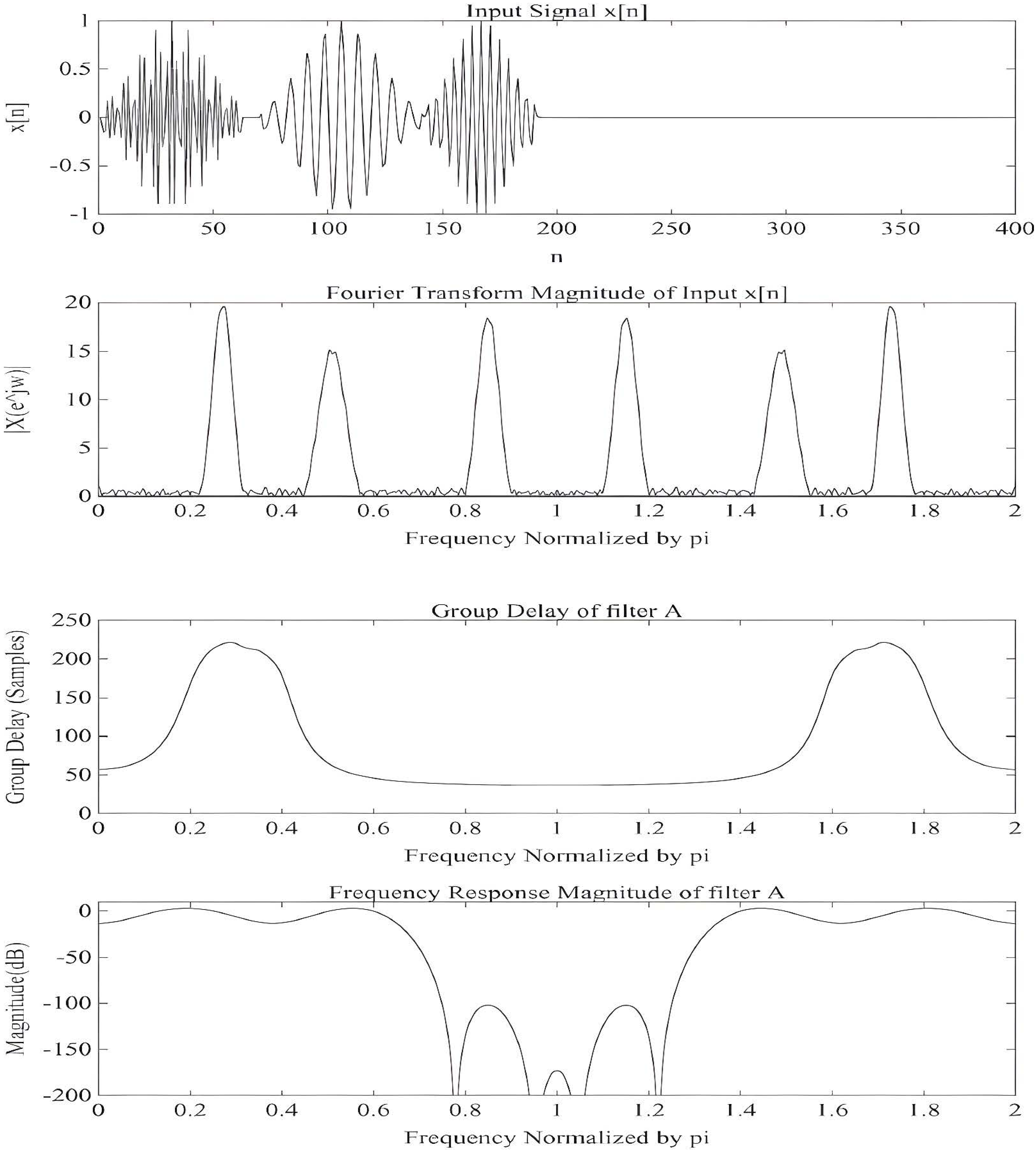
$\square$.

(we can also see from the definition of the inverse Fourier transform that $h[0] = 0$ in our case.)

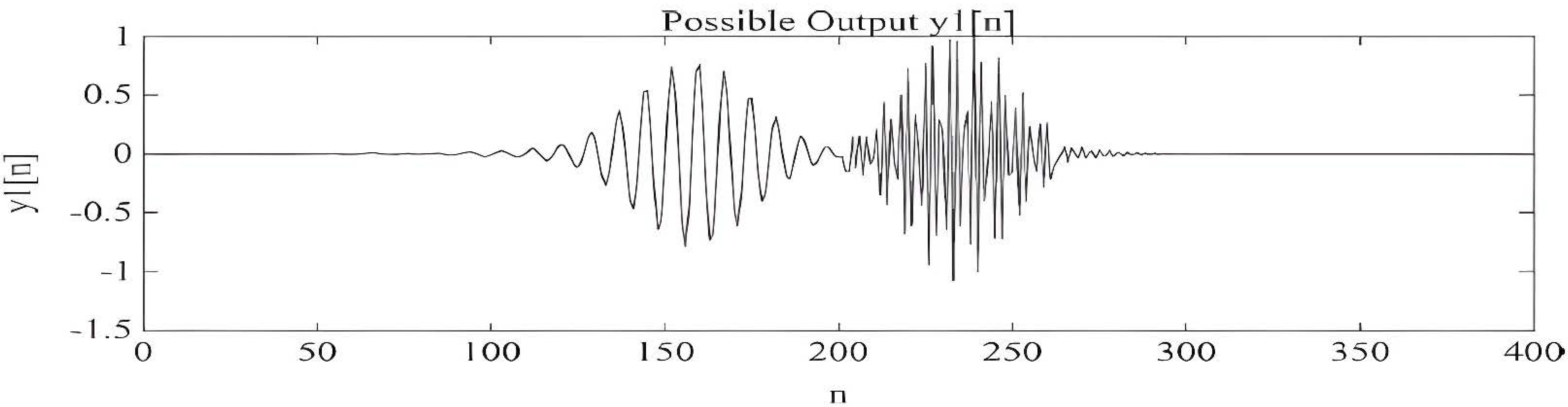
Exercise 5

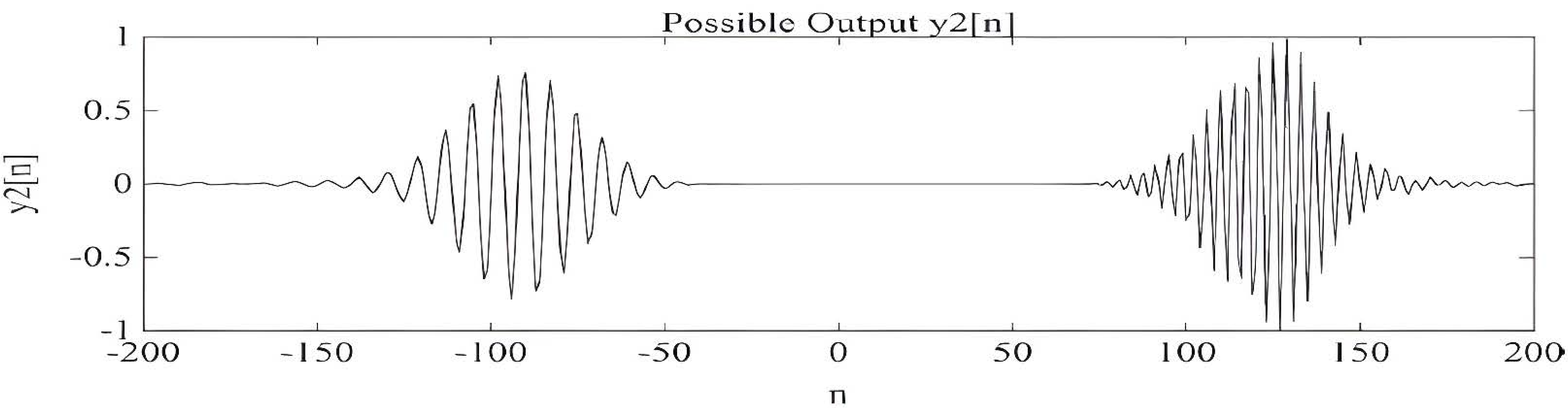
You are given the following plots. We have an input signal $x[n]$ that is filtered by filter A, and the task is to find which possible output $y_i[n]$, $i \in \{1, 2, \dots, 8\}$ is the correct one, and why.

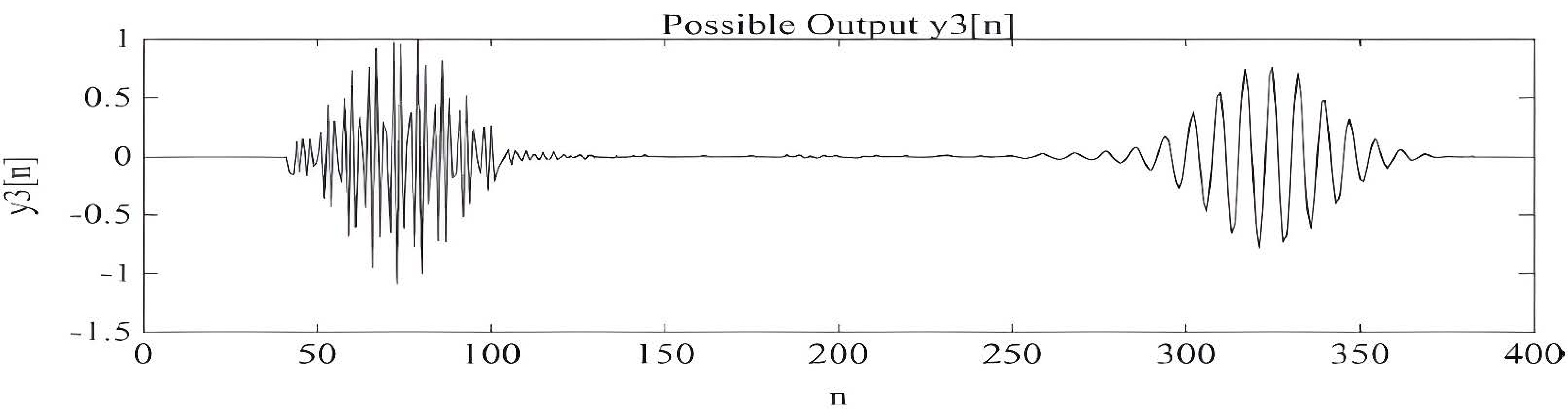
The first plot shows $x[n]$ in the time domain while the second plot shows the magnitude response of $x[n]$. Next, we have the filter's group delay plot followed by its magnitude response. Given this information, we should be able to determine what will happen to $x[n]$ after filtering it with A.

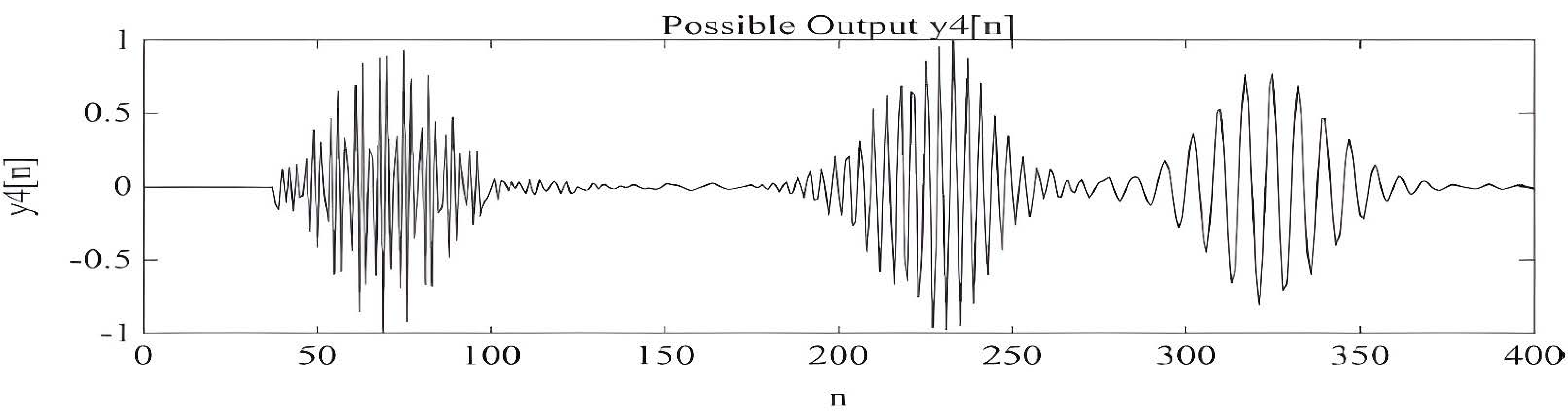


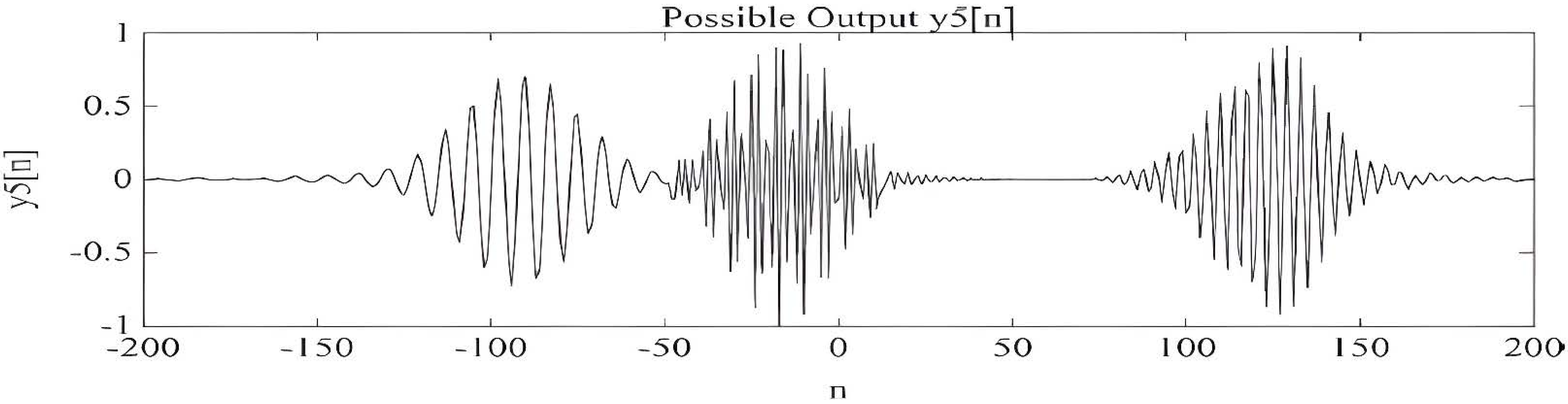
Input Signal x[n]
x[n]
n
1
0.5
0
-0.5
-1
0
50
100
150
200
250
300
350
400
Fourier Transform Magnitude of Input x[n]
|X(e^jw)|
Frequency Normalized by pi
20
15
10
5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Group Delay of filter A
Group Delay (Samples)
Frequency Normalized by pi
250
200
150
100
50
Frequency Response Magnitude of filter A
Magnitude(dB)
Frequency Normalized by pi
-50
-100
-150
-200

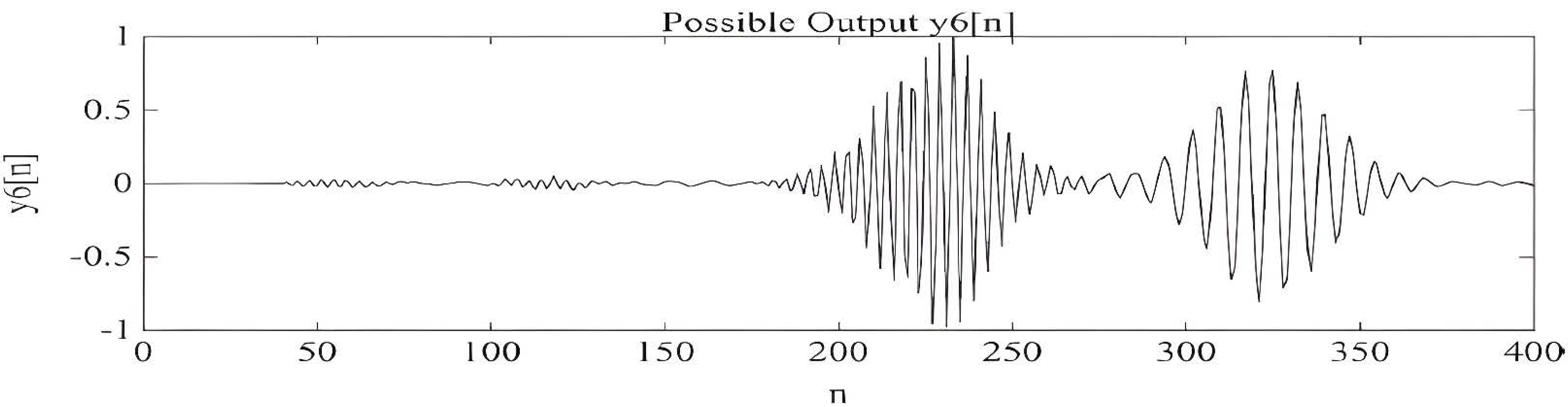

Possible Output y1[n]
y1[n]
n
1
0.5
0
-0.5
-1
-1.5
0
50
100
150
200
250
300
350
400

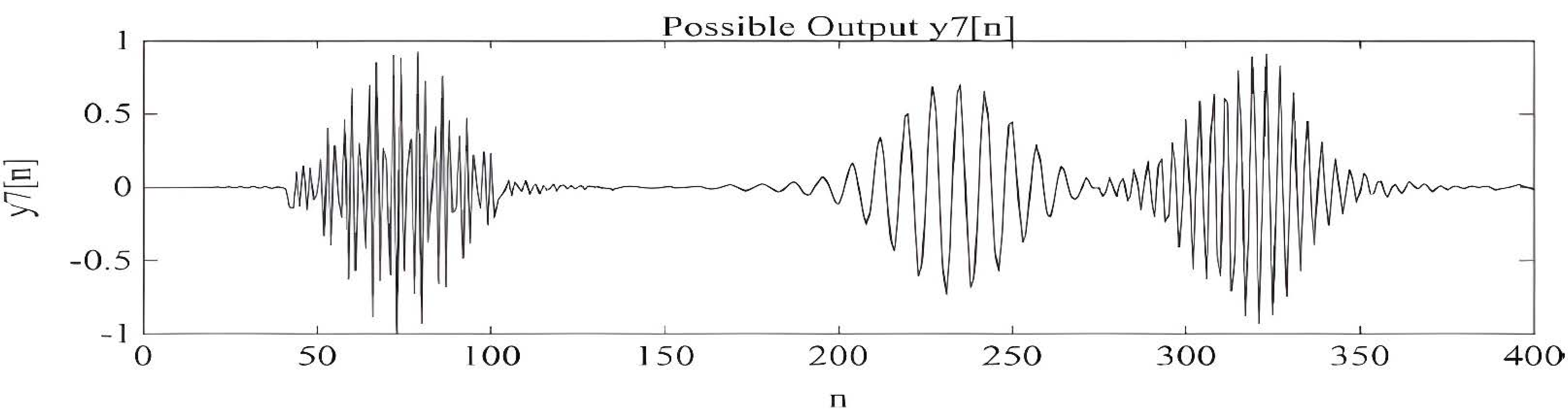

Possible Output y2[n]
y2[n]
n
1
0.5
0
-0.5
-1
-200
-150
-100
-50
0
50
100
150
200

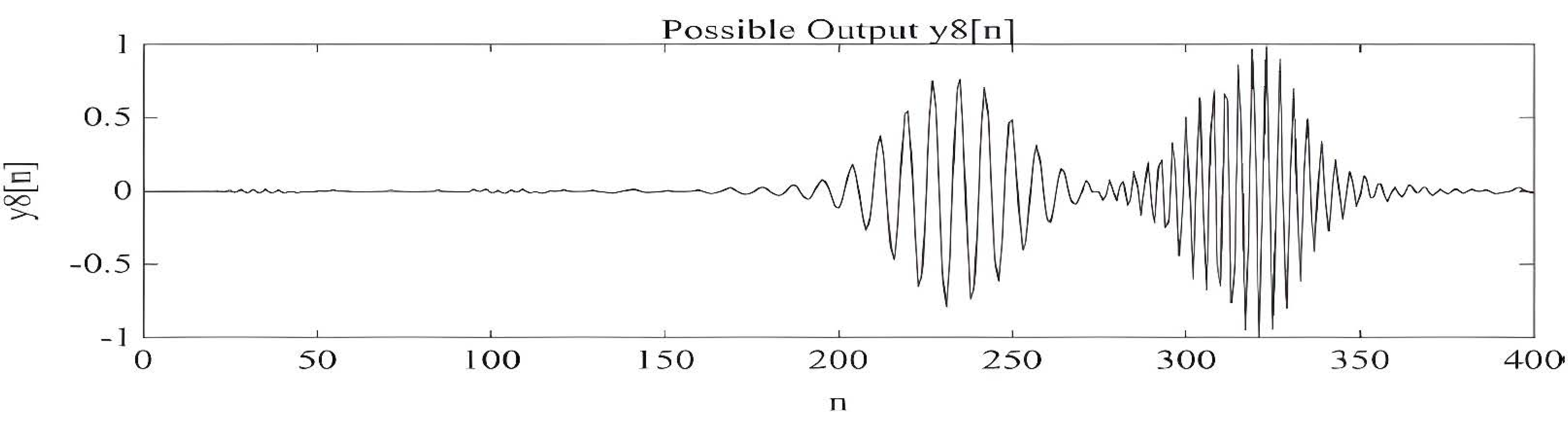

Possible Output y3[n]
y3[n]
n
1
0.5
0
-0.5
-1
-1.5
0
50
100
150
200
250
300
350
400


Possible Output y4[n]
y4[n]
n
1
0.5
0
-0.5
-1
0
50
100
150
200
250
300
350
400


Possible Output y5[n]
y5[n]
n
1
0.5
0
-0.5
-1
-200
-150
-100
-50
0
50
100
150
200


Possible Output y6[n]
y6[n]
n
1
0.5
0
-0.5
-1
0
50
100
150
200
250
300
350
400


Possible Output y7[n]
y7[n]
n
1
0.5
0
-0.5
-1
0
50
100
150
200
250
300
350
400


Possible Output y8[n]
y8[n]
n
1
0.5
0
-0.5
-1
0
50
100
150
200
250
300
350
400

About the dB scale: The decibel scale is a widely-used logarithmic scale for compressing large ranges of values and matching better with the human perception of sound and intensity. For a given magnitude spectrum say $|H(e^{j\omega})|$, converting it to dB is given by $\boxed{20\log_{10}|H(e^{j\omega})|}$. So, when $|H(e^{j\omega})| \to 0$, then $dB \to -\infty$ (attenuation), and when $|H(e^{j\omega})| \to +\infty$, then $dB \to +\infty$ (amplification).

Solution

1) From the magnitude spectrum of $x[n]$, we can determine that there are three (3) frequencies present in the signal: $\omega_1 = 0.25\pi$, $\omega_2 = 0.5\pi$, and $\omega_3 = 0.85\pi$ radians per sample.

2) The third (3rd) pulse of $x[n]$, which begins approximately at $n = 140$, corresponds to the middle frequency $\omega_2 = 0.5\pi$, as the other pulses have either higher or lower frequencies.

3) The magnitude spectrum of filter A shows that the frequencies $\omega_1 = 0.25\pi$ and $\omega_2 = 0.5\pi$ will remain unaffected (since $|H(e^{j\omega})| \to 1$ leads to $dB \to 0$). However, $\omega_3 = 0.85\pi$ will be significantly attenuated by approximately -100 dB. This translates to an amplitude reduction of $-100 = 20\log_{10}(x) \Leftrightarrow x = 10^{-5}$. This practically filters out the frequency ω_3 from $x[n]$ almost completely.

4) From the groupdelay of filter A we observe that the frequency $\omega_2 = 0.5\pi$ will experience a shift of approximately 50 samples, meaning we expect the 3rd pulse to start around $n = 190$.

5) The only outputs that satisfy condition 4) are $y_4[n]$ and $y_6[n]$.

6) Observing that $y_4[n]$ does not have any frequencies filtered out (it only shifted them), while $y_6[n]$ has one frequency filtered out as expected from 3), we conclude that $y_6[n]$ is the correct answer.

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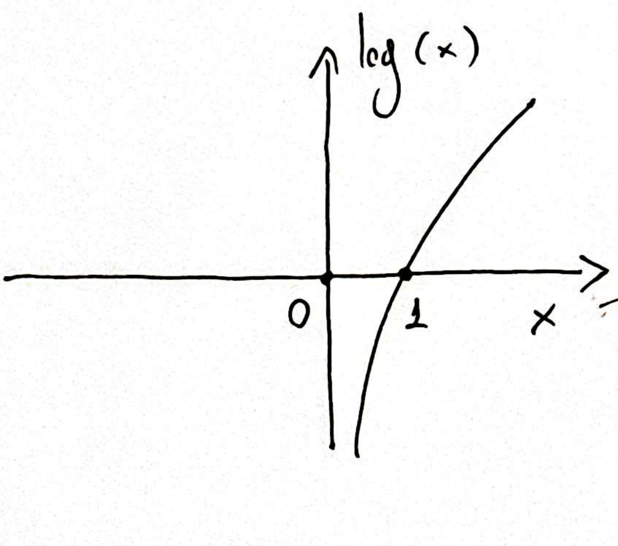
Another sanity check: We would also expect the lower frequency $\omega_1 = 0.25\pi$ to remain unaffected by A and only experience a shift of approximately 200 samples, which is indeed reflected at $y_6[n]$.



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