Φροντιστήριο 9º HY-370

Exercise 1

We are given the following impulse response:

$$h[n] = \delta[n] + 2\delta[n-1] - 2\delta[n-2] - \delta[n-3].$$

(α') What is the frequency response $H(e^{j\omega})$?

(β') Find and draw qualitatively the magnitude spectrum $|H(e^{j\omega})|$.

(γ') Find and draw qualitatively the phase spectrum $\measuredangle H(e^{j\omega})$.

Solution

(α') The frequency response definition is:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\omega n}$$, therefore, in our case:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \left(\delta[n] + 2\delta[n-1] - 2\delta[n-2] - \delta[n-3] \right) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} \left[\delta[n] e^{-j\omega n} \right] + 2 \sum_{n=-\infty}^{+\infty} \left[\delta[n-1] e^{-j\omega n} \right] - 2 \sum_{n=-\infty}^{+\infty} \left[\delta[n-2] e^{-j\omega n} \right] - \sum_{n=-\infty}^{+\infty} \left[\delta[n-3] e^{-j\omega n} \right]$$

↳ $\delta[n-n_0]$ is 0 except at $n=n_0$ where it is 1.

$$= \delta[n] e^{-j\omega n} \Big|_{n=0} + 2\delta[n-1] e^{-j\omega n} \Big|_{n=1} - 2\delta[n-2] e^{-j\omega n} \Big|_{n=2} - \delta[n-3] e^{-j\omega n} \Big|_{n=3}$$

$$= \boxed{1 + 2e^{-j\omega} - 2e^{-2j\omega} - e^{-3j\omega} = H(e^{j\omega})}$$

↳ This answer is correct, however, we will find difficulties if we attempt the following questions about magnitude and phase. We can simplify by finding out the sinusoids that are "hidden", having in mind:

$$\cos(x) = (e^{jx} + e^{-jx})/2, \quad \sin(x) = (e^{jx} - e^{-jx})/2j.$$

$$H(e^{j\omega}) = 1 - e^{-j3\omega} + 2e^{-j\omega} - 2e^{-j2\omega} =$$ they look like sines

$$= 2j \left(\frac{e^{j0\omega} - e^{-j3\omega}}{2j} + 2 \left(\frac{e^{-j\omega} - e^{-j2\omega}}{2j} \right) \right) =$$

↳ what to bring outside s.t. we have same powers on the inside?

$$= 2j\left(e^{-j\frac{3\omega}{2}}\left(\frac{e^{j\frac{3\omega}{2}} - e^{-j\frac{3\omega}{2}}}{2j}\right) + 2e^{-j\frac{3\omega}{2}}\left(\frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{2j}\right)\right)$$

$$= \boxed{2je^{-j\frac{3\omega}{2}}\left(\sin(3\omega/2) + 2\sin(\omega/2)\right) = H(e^{j\omega})}$$

(b') It will be:

$$|H(e^{j\omega})| = \left|2j\,e^{-j\frac{3\omega}{2}}\left(\sin(3\omega/2) + 2\sin(\omega/2)\right)\right| =$$

$$= |2|\cdot|j|\cdot\left|e^{-j3\omega/2}\right|\cdot\left|\sin(3\omega/2) + 2\sin(\omega/2)\right|$$

It is always the case that $|e^{jx}| = 1$, because:

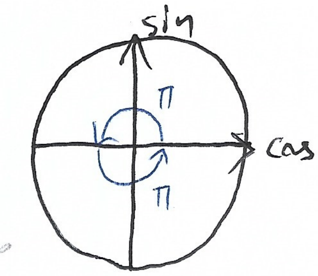
$$|e^{jx}| = |\cos(x) + j\sin(x)| = \sqrt{\cos^2(x) + \sin^2(x)} = \sqrt{1} = 1.$$

You can also claim that it expresses movement around the unit circle

$$= 2\cdot 1\cdot 1\cdot\left|\sin(3\omega/2) + 2\sin(\omega/2)\right| =$$

$$= \boxed{2\left|\sin(3\omega/2) + 2\sin(\omega/2)\right| = |H(e^{j\omega})|}$$

↳ Tips on how to draw functions like these:

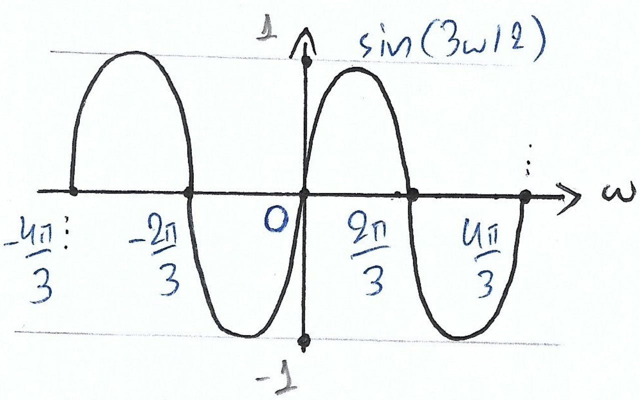


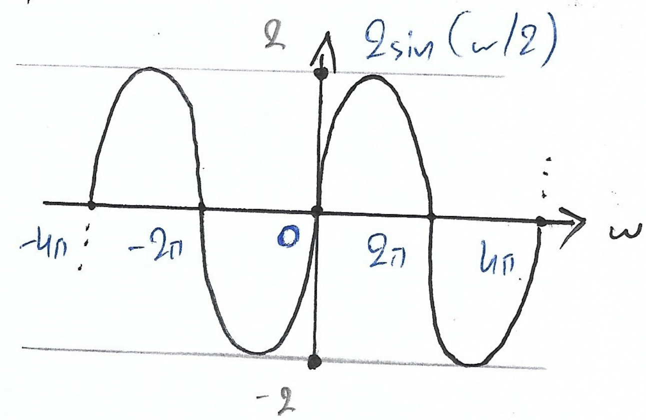
↳ Let us first find where each sine is zero:

- $\sin(3\omega/2) = 0 \Leftrightarrow \frac{3\omega}{2} = k\cdot\pi \Leftrightarrow \omega = 2k\pi/3 \quad , k\in\mathbb{Z}.$

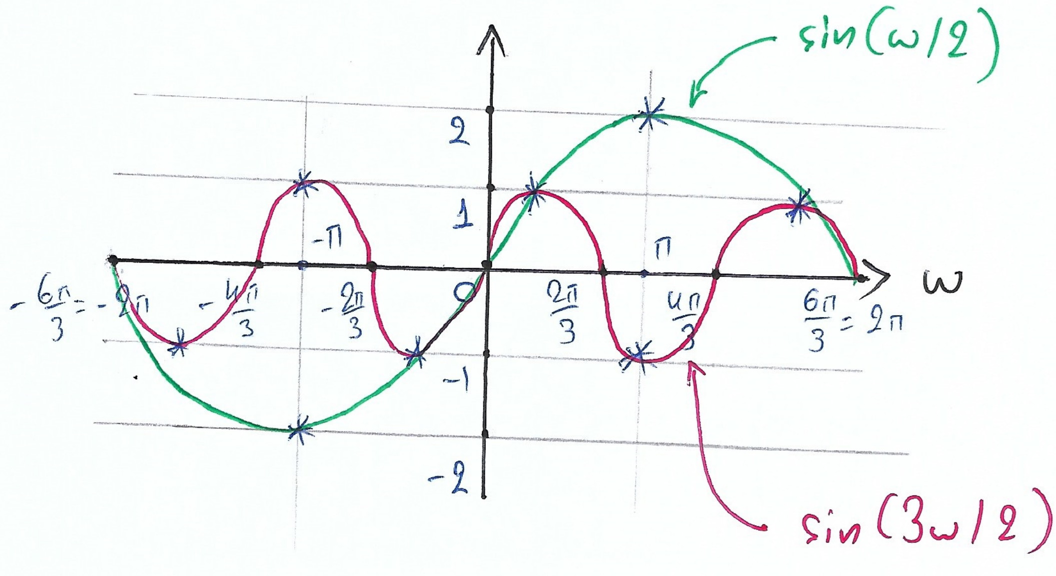
- $\sin(\omega/2) = 0 \Leftrightarrow \frac{\omega}{2} = k\cdot\pi \Leftrightarrow \omega = 2k\pi \quad , k\in\mathbb{Z}.$

↳ Let us draw them individually at first:



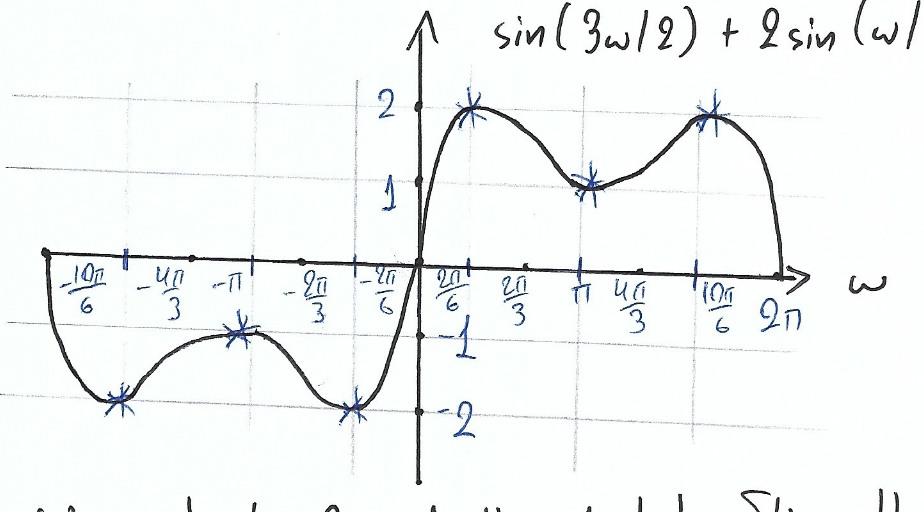


↳ Now, let us just put them in the same plot:

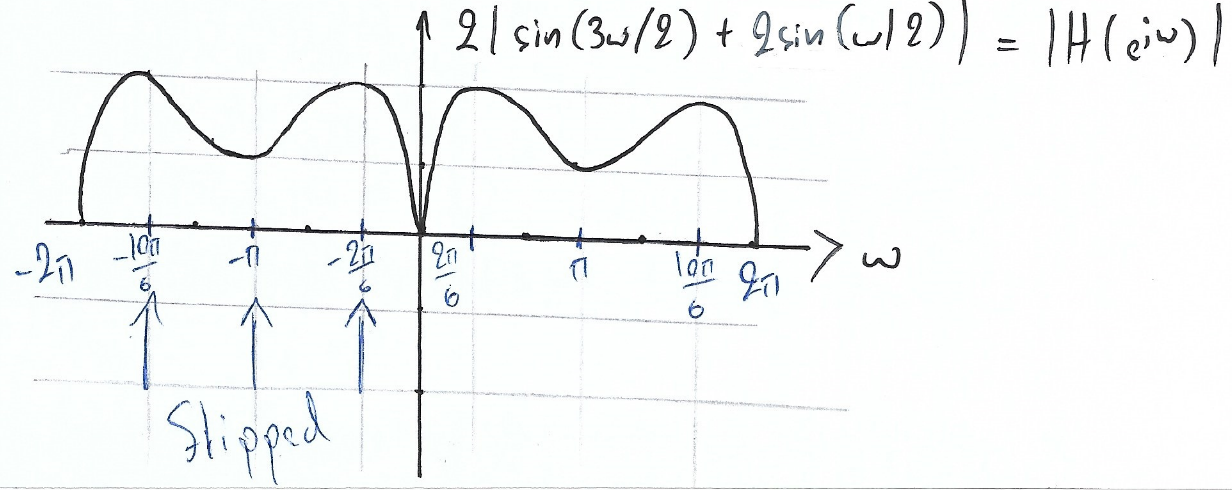


$*$ = local minima / maxima

↳ Combine: Think of what happens when we add the green and red graphs. It is useful to consider what will happen to the minima and maxima (noted as * in the plot). For instance, we have these 3 bumps on each side, and for $\omega = \pi$ the red bump will come at +1 since +2 will be added to −1. Similarly, the other two bumps (left and right of $\omega = \pi$) will go up for $\omega > 0$, and down for $\omega < 0$, since we are adding numbers of the same sign. Hence, in total:



↳ Finalizing: We scale by 2 and the absolute flips the left side:



(c') Multiplication $\rightarrow$ addition of separate phases:

$$\measuredangle H(e^{j\omega}) = \measuredangle 2 + \measuredangle j + \measuredangle e^{-j3\omega/2} + \measuredangle\left(\sin\left(\frac{3\omega}{2}\right) + 2\sin\left(\frac{\omega}{2}\right)\right)$$

$$= \measuredangle 2e^{j0} + \measuredangle e^{j\pi/2} + \measuredangle e^{-j3\omega/2} + \measuredangle\left(\sin\left(\frac{3\omega}{2}\right) + 2\sin\left(\frac{\omega}{2}\right)\right)$$

↳ phase is expressed in $[-\pi, \pi]$ ↳ $\underline{\measuredangle e^{j\varphi} = \varphi}$

↳ How to determine the phase of a real function?

- If positive then $\varphi = 0$ $\forall\omega$.
- Else if negative and $\omega > 0$ then $\varphi = \pi$ (by convention).
- Else if negative and $\omega < 0$ then $\varphi = -\pi$ (also by convention).
- Else, function is zero so the phase is not defined.

(there is no exponent that gives zero).

- The sinusoid sum: $\sin\left(\frac{3\omega}{2}\right) + 2\sin\left(\frac{\omega}{2}\right)$ is a real function and in $(0, \pi]$ it is positive, so $\varphi = 0$, and in $[-\pi, 0)$ it is negative, so $\varphi = -\pi$. For $\omega = 0$ the phase is undefined.
- Alternatively, if we draw in $(0, \pi]$ we can also draw in $[-\pi, 0)$ because the signal is real, so we have an odd symmetry!

Therefore, in total for the phase spectrum:

$$\measuredangle H(e^{j\omega}) = \begin{cases} 0 + \frac{\pi}{2} - \frac{3\omega}{2} + 0 \,, & \omega \in (0,\pi] \\ 0 + \frac{\pi}{2} - \frac{3\omega}{2} - \pi \,, & \omega \in [-\pi,0) \end{cases} =$$

$$= \begin{cases} -\frac{3\omega}{2} + \frac{\pi}{2} \,, & \omega \in (0,\pi] \\ -\frac{3\omega}{2} - \frac{\pi}{2} \,, & \omega \in [-\pi,0) \end{cases}$$

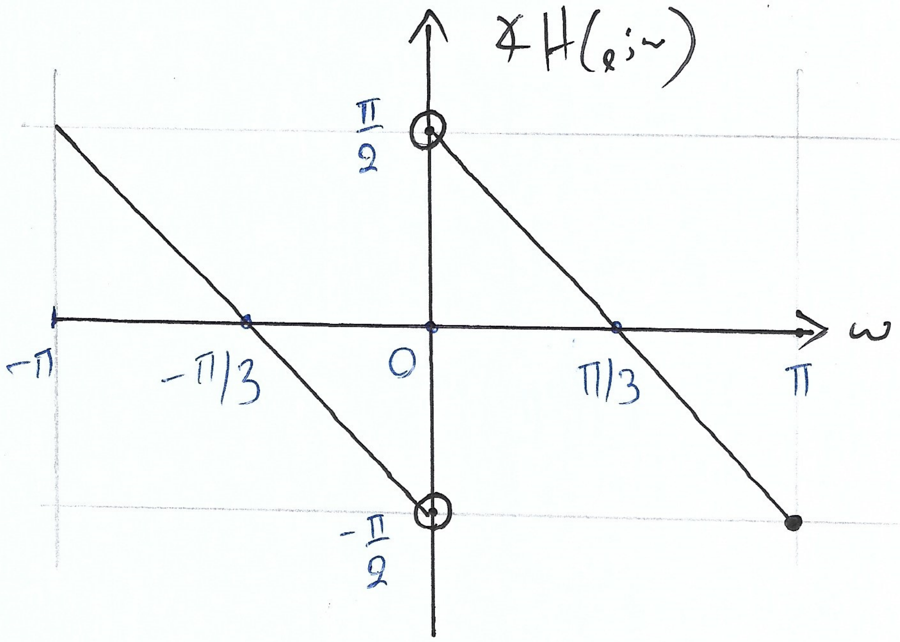
notice the odd symmetry

$\hookrightarrow \measuredangle H(e^{j\omega}) = 0 \Rightarrow -\frac{3\omega}{2} \pm \frac{\pi}{2} = 0 \Leftrightarrow \omega = \pm\pi/3$

$\hookrightarrow \measuredangle H(e^{j\omega}) = \pm\frac{\pi}{2} \Rightarrow -\frac{3\omega}{2} \pm \frac{\pi}{2} = \pm\frac{\pi}{2} \Leftrightarrow \omega = 0$

Find two points to draw the lines

not defined, but we just need the point.



Exercise 2

An LTI system is described by the following difference equation:

$$y[n] = x[n] + x[n-10].$$

(a') Find and draw the amplitude and phase responses of the system, using:

$$e^{ja} + e^{-jb} = e^{-j(a-b)/2} \cdot \left(e^{j(a+b)/2} + e^{-j(a+b)/2} \right), \quad (i).$$

(b') Find the output of the system for inputs:

I) $x[n] = \cos\left(\frac{\pi n}{10}\right) + 3\sin\left(\frac{\pi n}{3} + \frac{\pi}{10}\right)$

II) $x[n] = 10 + 5\cos\left(\frac{2\pi n}{5} + \frac{\pi}{10}\right)$

Solution

(a') First, we need to find the frequency response of the system. Its impulse response is:

$$h[n] = \delta[n] + \delta[n-10].$$

So by definition of the frequency response, we have:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} \left(\delta[n] + \delta[n-10]\right) e^{-j\omega n} =$$

$$= \sum_{n=-\infty}^{+\infty} \delta[n]\, e^{-j\omega n} + \sum_{n=-\infty}^{+\infty} \delta[n-10]\, e^{-j\omega n} =$$

$$= \delta[n]\, e^{-j\omega n}\Big|_{n=0} + \delta[n-10]\, e^{-j\omega n}\Big|_{n=10} =$$

$$= \boxed{1 + e^{-j10\omega} = H(e^{j\omega})} \overset{(i)}{\Longleftrightarrow}$$ using the given formula

$$\Leftrightarrow H(e^{j\omega}) = e^{-j0\omega} + e^{-j10\omega} = e^{j(-10\omega)/2}\left(e^{j(10\omega)/2} + e^{-j(10\omega)/2}\right) =$$

$$= 2e^{-j5\omega}\left(e^{j5\omega} + e^{-j5\omega}\right)/2$$

$$= \boxed{2e^{-j5\omega}\cos(5\omega) = H(e^{j\omega})}$$

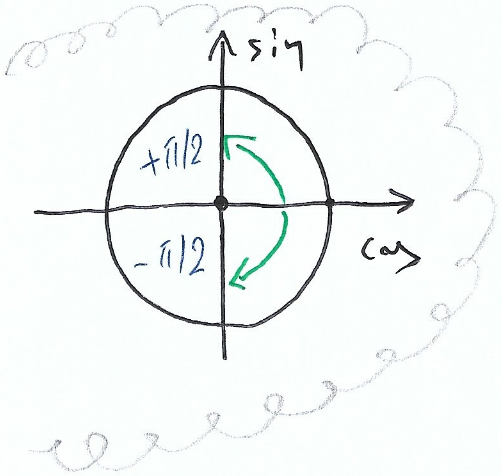
This is a more convenient form to calculate the magnitude and phase spectra.

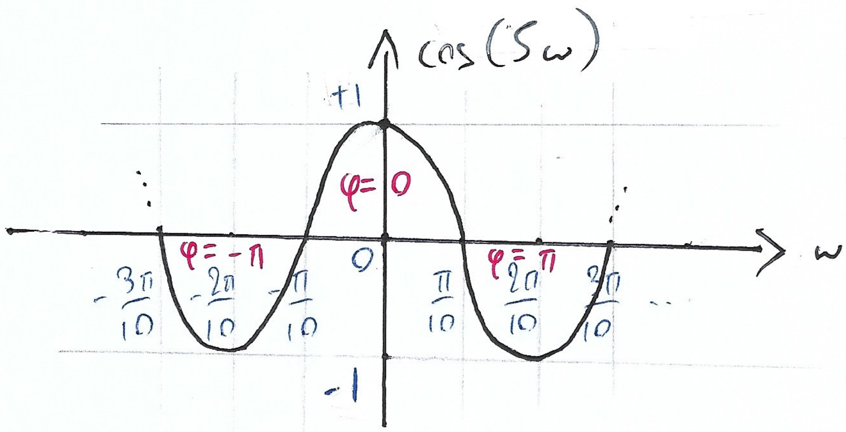
So, for the magnitude response of the system:

$$|H(e^{j\omega})| = |2|\,|e^{-j5\omega}|\,|\cos(5\omega)| = \boxed{2|\cos(5\omega)| = |H(e^{j\omega})|}$$

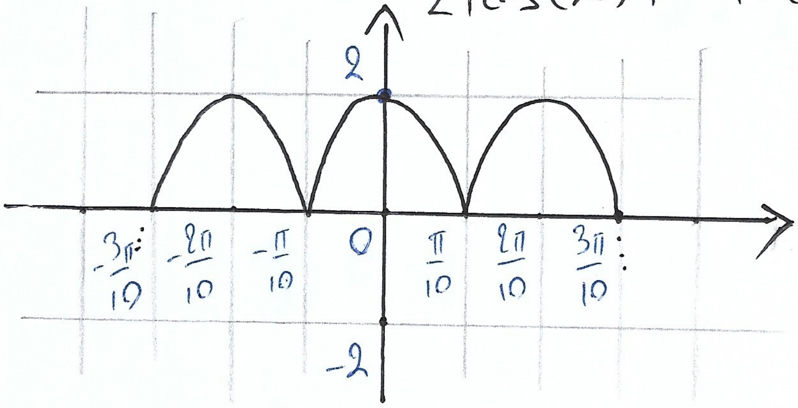
$\hookrightarrow$ To draw it, we find: $\cos(5\omega) = 0 \iff 5\omega = 2k\pi \pm \frac{\pi}{2} \iff$

$$\iff \omega = \frac{2k\pi}{5} \pm \frac{\pi}{10}$$





scale by 2 and flip negatives



- And for the phase response:

$$\measuredangle H(e^{j\omega}) = \measuredangle 2 + \measuredangle e^{-j5\omega} + \measuredangle \cos(5\omega) = \boxed{-5\omega + \measuredangle\cos(5\omega) = \measuredangle H(e^{j\omega})}$$

$$\measuredangle \cos(5\omega) = \begin{cases} 0, & \cos(5\omega) > 0 \\ \pi, & \cos(5\omega) < 0,\ \omega > 0 \\ -\pi, & \cos(5\omega) < 0,\ \omega < 0 \end{cases}$$

↳ We found that its sign changes every $\frac{2k\pi}{5} \pm \frac{\pi}{10}$:

So we either add or subtract π from -5ω every time till $\omega = \pm\pi$

$$= \begin{cases} \vdots \\ -\pi, & -3\pi/10 < \omega < -\pi/10 \\ 0, & -\pi/10 < \omega < \pi/10 \\ \pi, & \pi/10 < \omega < 3\pi/10 \\ \vdots \end{cases}$$

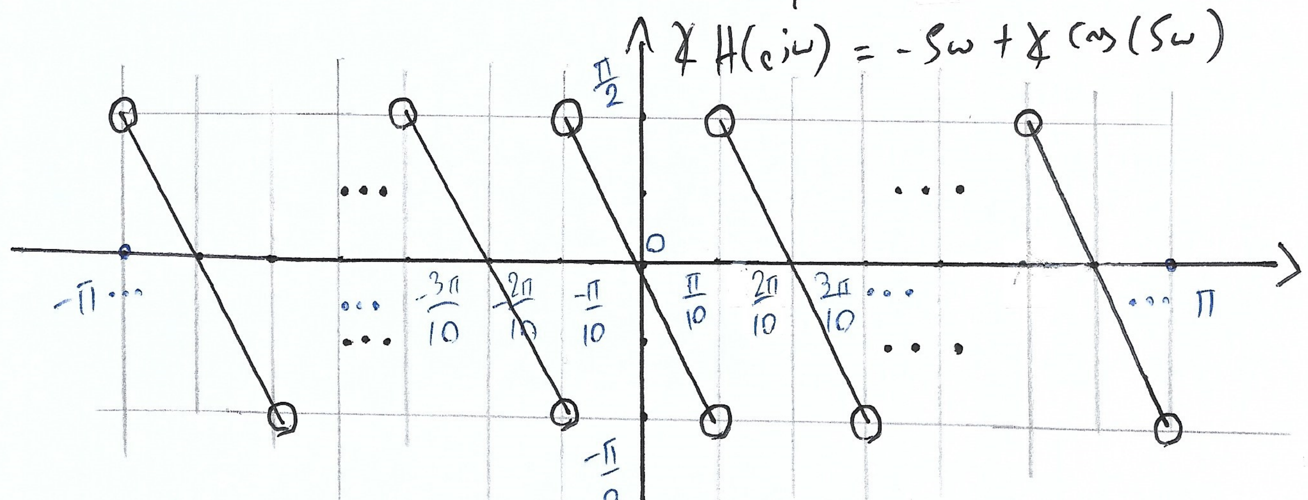
Not $\leq$ because it is undefined every $\frac{2k\pi}{5} \pm \frac{\pi}{10}$!

↳ this behavior continues until $\omega = \pm\pi$.

↳ To draw it we find two points again:

- $\measuredangle H(e^{j\omega})\big|_{\omega=0} = 0$
- $\measuredangle H(e^{j\omega})\big|_{\omega=\pm\frac{\pi}{10}} = \pm\frac{5\pi}{10} = \pm\frac{\pi}{2}$

(we don't care if its not defined: we just need the points of the line)



(b') We know from the course's theory what is the form of the output given sinusoid sums as inputs to an LTI system? Reminder: For input:

$$x[n] = \sum_{k=1}^{N} A_k \cdot \cos(\omega_k n + \vartheta_k)$$

, the output is:

$$y[n] = \sum_{k=1}^{N} A_k \cdot |H(e^{j\omega_k})| \cos(\omega_k n + \vartheta_k + \measuredangle H(e^{j\omega_k}))$$

For I) we can see that $\omega_k = \{\frac{\pi}{10}, \frac{\pi}{3}\}$, $A_k = \{1, 3\}$, $\vartheta_k = \{0, \frac{\pi}{10}\}$, $k = 1, 2$, thus:

$$Y_I[n] = 1 \cdot |H(e^{j\pi/10})| \cdot \cos\left(\frac{\pi n}{10} + 0 + \measuredangle H(e^{j\pi/10})\right) +$$

$$3|H(e^{j\pi/3})| \cdot \sin\left(\frac{\pi n}{3} + \frac{\pi}{10} + \measuredangle H(e^{j\pi/3})\right)$$

zero!

$$= 2 \cdot \left|\cos\left(5\frac{\pi}{10}\right)\right| \cos\left(\frac{\pi n}{10} + \measuredangle H(e^{j\pi/10})\right) +$$

1/2

$$3 \cdot 2 \cdot |\cos(5\pi/3)| \cdot \sin\left(\frac{\pi n}{3} + \frac{\pi}{10} - \frac{5\pi}{3}\right)$$

$$= \boxed{3 \sin\left(\frac{\pi n}{3} - \frac{47\pi}{30}\right) = Y_I[n]}$$

- We also know the output formula in case of complex exponential series as inputs to an LTI system:

If $x[n] = \sum_{k=1}^{N} A_k e^{j(\omega_k n + \theta_k)}$, then the output is:

$$y[n] = \sum_{k=1}^{N} A_k H(e^{j\omega_k}) e^{j(\omega_k n + \theta_k)}.$$

↳ For II), for instance, we can treat 10 as $10e^{j0}$, so:

$$y[n] = 10\, H(e^{j0})\, e^{j0} + 5\left|H(e^{j2\pi/5})\right| \cos\left(\frac{2\pi n}{5} + \frac{\pi}{2} + \measuredangle H(e^{j2\pi/5})\right)$$

$$= 10 \cdot 2 \cdot 1 + 5 \cdot 2 \cdot \left|\cos\left(5 \cdot \frac{2\pi}{5}\right)\right| \cos\left(\frac{2\pi n}{5} + \frac{\pi}{2} - 5 \cdot \frac{2\pi}{5} + 0\right)$$

$$= \boxed{20 + 10\cos\left(\frac{2\pi n}{5} + \frac{\pi}{2}\right) = y[n]}$$

END