Solutions for the first set 2020

1^{o} Φροντιστήριο HY-370

Exercise 1

Given the signal: $h[n] = \left(\frac{1}{2}\right)^{n-1} \left(u[n+3] - u[n-10]\right)$

Find the relationship of A, B, n, such that the following holds:

$$h[n-k] = \begin{cases} \left(\frac{1}{2}\right)^{n-k-1}, & A \leq k \leq B \\ 0, & \text{elsewhere.} \end{cases}$$

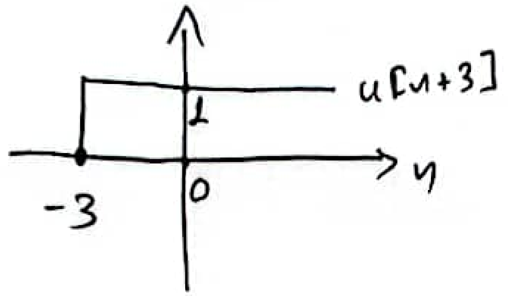
Solution

↳ Essentially we're asked to find where $h[n-k]$ is not zero.

↳ We get $h[n-k]$ if we shift $h[-k]$ by n, or $h[n]$ by $-k$.

↳ So, where is $h[n]$ equal to zero? This is answered from:

$$u[n+3] = \begin{cases} 1, & n \geq 3 \\ 0, & \text{elsewhere} \end{cases} \qquad -u[n-10] = \begin{cases} -1, & n \geq 10 \\ 0, & \text{elsewhere} \end{cases}$$



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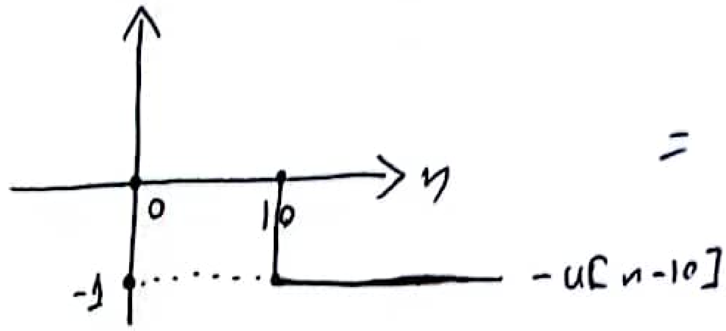

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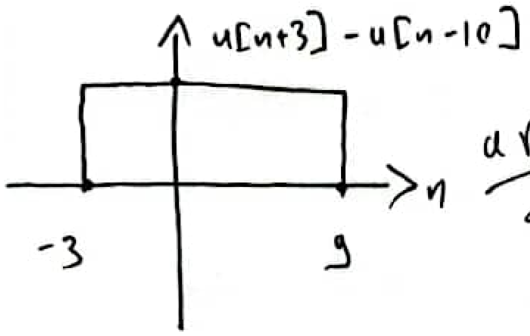


a rect!

Also algebraically:

$$u[n+3]-u[n-10] = \begin{cases} 1, & -3 \le n \le 9 \\ 0, & \text{elsewhere}. \end{cases}$$

↳ Now if we shift by $-k$ we get:

$$u[n-k+3]-u[n-k-10] = \begin{cases} 1, & -3 \le n-k \le 9 \\ 0, & \text{elsewhere} \end{cases} =$$

$$= \begin{cases} 1, & -3-n \le -k \le 9-n \\ 0, & \text{elsewhere} \end{cases} = \begin{cases} 1, & n-9 \le k \le n-3 \\ 0, & \text{elsewhere} \end{cases},$$

hence: $\boxed{A = n-9,\ B = n+3.}$

↳ In short, since $h[n]$ is non-zero only where $-9 \le n \le 3$, then shifting $h[-k]$ by n, i.e., constructing $h[n-k]$, this will be non-zero at $n-9 \le k \le n+3$.

Exercise 2

Three LTI systems with impulse responses $h_1[n]$, $h_2[n]$, and $h_3[n]$ are connected in series. We know that:

$$h_2[n] = u[n] - u[n-2], \quad \text{and that}$$

$$h[n] = (h_1 * h_2 * h_2)[n] =$$

$$= \begin{cases} 0, & n \leq -1 \text{ and } n \geq 7 \\ 1, & n = 0 \\ 5, & n = 1 \\ 10, & n = 2 \\ 11, & n = 3 \\ 8, & n = 4 \\ 4, & n = 5 \\ 1, & n = 6. \end{cases}$$

In series: $h_1 * h_2 * \cdots$

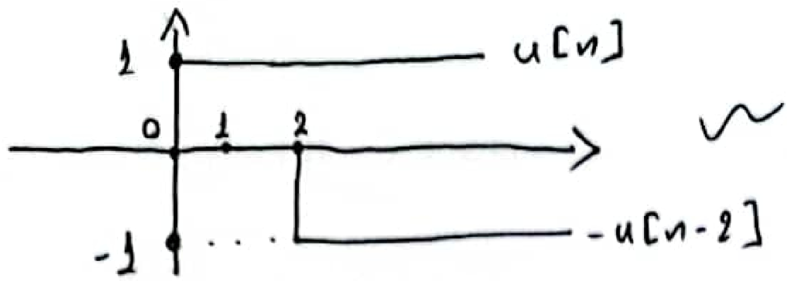
In parallel: $h_1 + h_2 + \cdots$

Find the impulse response $h_1[n]$.

Solution

↳ We can start by calculating $(h_2 * h_2)[n]$ because it's easy:

$u[n] - u[n-2] = \delta[n] + \delta[n-1]$, (you can draw them to confirm it :)



$\leadsto$ They cancel out except at $n=0$ and $n=1$.

So: $(h_2 * h_2)[n] = (\delta[n] + \delta[n-1]) * (\delta[n] + \delta[n-1])$

$\delta[n-n_1] * \delta[n-n_2] = \delta[n-n_1-n_2]$

property

$= \delta[n] + \delta[n-1] + \delta[n-1] + \delta[n-2]$

$= \delta[n] + 2\delta[n-1] + \delta[n-2]$.

Hence: $h[n] = h_1[n] * (h_2 * h_2)[n] =$

$= h_1[n] * (\delta[n] + 2\delta[n-1] + \delta[n-2])$

$= h_1[n] + 2h_1[n-1] + h_1[n-2]$.

Now we can just plug in the values that are given:

- $1 = h_1[0] + 2h_1[0-1] + h_1[0-2] \Leftrightarrow$

$\Leftrightarrow 1 = h_1[0] - 2h_1[-1] + h_1[-2] \Rightarrow$

$\Rightarrow 1 = h_1[0] - 2 \cdot 0 + 0 \Leftrightarrow \boxed{h_1[0] = 1}$

(8) We assume that an LTI is causal unless otherwise stated, so $\forall n<0$ $h[n]=0$

- $5 = h_1[1] + 2h_1[0] + h_1[-1] \Rightarrow \boxed{h_1[1] = 3}$
- $10 = h_1[2] + 2h_1[1] + h_1[0] \Rightarrow \boxed{h_1[2] = 3}$
- $11 = h_1[3] + 2h_1[2] + h_1[1] \Rightarrow \boxed{h_1[3] = 2}$
- $8 = h_1[4] + 2h_1[3] + h_1[2] \Rightarrow \boxed{h_1[4] = 1}$
- $4 = h_1[5] + 2h_1[4] + h_1[3] \Rightarrow \boxed{h_1[5] = 0}$
- $1 = h_1[6] + 2h_1[5] + h_1[4] \Rightarrow \boxed{h_1[6] = 0}$

↳ Observe that for all other possible values of n, $h_1[n]$ will be zero. So, in total, the answer is:

$$\boxed{h_1[n] = \begin{cases} 1, & n=0 \\ 3, & n=1 \text{ and } n=2 \\ 2, & n=3 \\ 1, & n=4 \\ 0, & \text{elsewhere.} \end{cases}}$$

Exercise 3

Show if the following systems are (a) linear, (b) stable (c) causal, (d) time-invariant, (e) dynamic:

i. $y[n] = x[4n+1]$, ii. $y[n] = (n+1)\,x[n]$,

iii. $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1. \end{cases}$

Solution

Reminders: For a system to be linear*, it needs to be homogeneous, i.e.:

if $x[n] \to y[n]$ then $c \cdot x[n] \to c \cdot y[n]$,

and additive, i.e.:

if $x_1[n] \to y_1[n]$ and $x_2[n] \to y_2[n]$, then

$x_1[n] + x_2[n] \to y_1[n] + y_2[n]$.

*linearity is also called superposition principle

tip: We can check for both at the same time if we want: $c_1 x_1[n] + c_2 x_2[n] \to c_1 y_1[n] + c_2 y_2[n]$.

(a) To check for linearity, we input $c_1 x_1[n] + c_2 x_2[n]$:

i. The output is: $c_1 x_1[4n+1] + c_2 x_2[4n+1] =$

$= c_1 y_1[n] + c_2 y_2[n]$, so it is linear.

ii. The output will be: $c_1 (n+1) x_1[n] + c_2 (n+1) x_2[n] =$

$= c_1 y_1[n] + c_2 y_2[n]$, so it is linear.

iii. There is an alternative way to write this system:

$$y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases} = x[n]u[n-1] + x[n]u[-n-1] + 0\,\delta[n]$$

Now the output is: $c_1 x_1[n] u[n-1] + c_1 x_1[n] u[-n-1] +$

$c_2 x_2[n] u[n-1] + c_2 x_2[n] u[-n-1] =$

$= c_1 y_1[n] + c_2 y_2[n]$, so it is linear.

(b) Bounded Input Bounded Output (BIBO) stability holds if:

$|x[n]| < B_x$ then $|y[n]| < B_y$,

i.e., we do not get infinity by feeding a bounded input to the system.

i. If $|x[n]| < B_x$, then $|y[n]| = |x[4n+1]| < B_x$, so the system is stable.

ii. If $|x[n]| < B_x$, then $|y[n]| = |(n+1)\,x[n]| =$

$= |n+1|\,|x[n]| < (n+1)B_x$.

This system is not stable because for $n \to \infty$, $|y[n]| \to \infty$.

iii. If $|x[n]| < B_x$, then $|y[n]| = |x[n]\,u[n-1] + x[n]\,u[-n-1]| = |x[n]\,(u[n-1] + u[-n-1])| = |x[n]| < B_x$, so this system is stable.

(c) One rule to determine causality, is that if the output depends on future values of the input, then this is not a causal system, otherwise it is.

i. We have dependency on future input values, so the system is not causal, e.g., $y[0] = x[1]$.

ii. We do not have dependency on future input values, so this is a causal system.

iii. Exactly the same as before, so this is a causal system.

(d) For time-invariance to hold, then delaying the input by n_0 samples should also delay the output by n_0 samples, i.e., for input $x[n-n_0]$ we should get output $y[n-n_0]$, else the system is not time-invariant.

i. For $x[n-n_0]$, we get $T\{x[n-n_0]\} = x[4n+1-n_0]$,

$\neq$

but $y[n-n_0] = x[4(n-n_0)+1] = x[4n-4n_0+1]$ is different, so this is not time-invariant.

ii. For $x[n-n_0]$, the output is $T\{x[n-n_0]\} = (n+1)\,x[n-n_0]$,

but $y[n-n_0] = (n-n_0+1)\,x[n-n_0] \neq T\{x[n-n_0]\}$,

hence not time-invariant.

iii. For $x[n-n_0]$, the output we get is $T\{x[n-n_0]\} =$

$= x[n-n_0]u[n-1] + x[n-n_0]u[-n-1]$, different than:

$y[n-n_0] = x[n-n_0]\,u[n-n_0-1] + x[n-n_0]u[-n-n_0-1]$,

so also not time-invariant.

(e) If a system needs to store/remember past or future values of the input, then it is dynamic.

i. This is a dynamic system, e.g., $y[1]$ needs $x[5]$.

ii. Only current values of the input are needed, so this is not a dynamic system.

iii. Same as before, so not dynamic.

Exercise 4

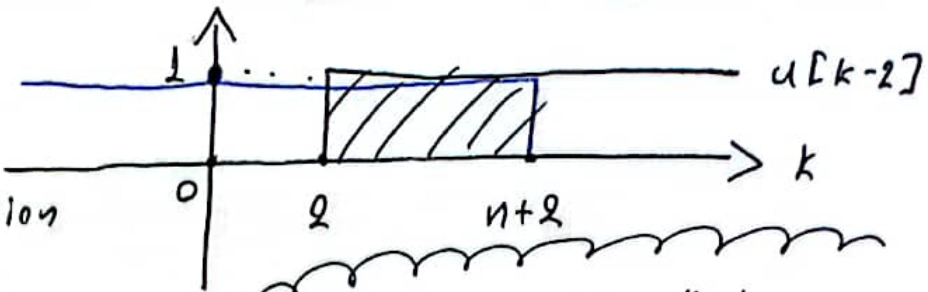
Convolve $x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2]$, with $y[n] = u[n+2]$.

Solution

Algebraic solution via convolution's definition:

$$c[n] = x[n] * y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, y[n-k] =$$

$$= \sum_{k=-\infty}^{+\infty} \left(\frac{1}{2}\right)^{k-2} \underbrace{u[k-2]\, u[n+2-k]}_{\text{unit steps determine the summation bounds}}$$



↳ so the convolution is not zero only where $2 \le k \le n+2$.

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

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$$= \sum_{k=2}^{n+2} \left(\frac{1}{2}\right)^{k-2} = \left(\frac{1}{2}\right)^{-2} \cdot \sum_{k=2}^{n+2} \left(\frac{1}{2}\right)^{k} =$$

$$= 4 \cdot \frac{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^{n+3}}{1-\frac{1}{2}} = 4 \cdot \frac{\frac{1}{4} - \frac{1}{8}\left(\frac{1}{2}\right)^{n}}{\frac{1}{2}} =$$

$$= 2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right), \quad n \geq 0.$$

Exercise 5

We are given the following difference equation and initial conditions:

$$y[n] - \frac{1}{4} y[n-1] = -2x[n], \quad y[-1] = 2.$$

Let an input signal be $x[n] = \left(\frac{1}{3}\right)^{n-1} u[n-1]$. Find (a) the zero input response $y_{zi}[n]$, (b) the impulse response $h[n]$, (c) the zero state response $y_{zs}[n]$, (d) whether it is stable, (e) its total output y_{T}[...]

(a) Similarly to continuous time, we find the characteristic polynomial: $\lambda - \frac{1}{4} = 0 \Leftrightarrow \lambda = 1/4$. Hence, the zero input response has the following form:

$$y_{zi}[n] = c \cdot \left(\frac{1}{4}\right)^n, \ n \geq 0.$$

see theory for more details as to why

For $y[-1] = 2$: $\quad 2 = c \cdot \left(\frac{1}{4}\right)^{-1} \Leftrightarrow c = 1/2$, so:

$$\boxed{y_{zi}[n] = \frac{1}{2}\left(\frac{1}{4}\right)^n u[n].}$$

removed coefficient!

(b) For the impulse response of $y[n] - \frac{1}{4}y[n-1] = x[n]$, we set $x[n] = \delta[n]$ and $y[n] = h_0[n]$:

$$h_0[n] - \frac{1}{4} h_0[n-1] = \delta[n].$$

From our theory we know that the solution is of the following form: $h_0[n] = c\left(\frac{1}{4}\right)^n, \ n \geq 0$. We can find the constant c by plugging in $n = 0$:

$$h_0[0] - \cancel{\frac{1}{4} h_0[-1]} = 1, \text{ hence } c = 1.$$

causal!

So, for our system: $y[n] - \frac{1}{4}y[n-1] = -2x[n]$, the

coeffient is back

impulse response is: $h[n] = -2h_0[n] = \boxed{-2\left(\frac{1}{4}\right)^n u[n] = h[n]}$

(c) The zero state response is given by convolving the input with the impulse response of the system:

$$y_{zs}[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k] =$$

$$= \sum_{k=-\infty}^{+\infty} \left(\frac{1}{3}\right)^{k-1} u[k-1] \left(-2\left(\frac{1}{4}\right)^{n-k} u[n-k]\right)$$

$$= \left(\frac{1}{3}\right)^{-1}(-2)\left(\frac{1}{4}\right)^n \sum_{k=-\infty}^{+\infty} \left(\frac{1}{3}\right)^k \left(\frac{1}{4}\right)^{-k} \underbrace{u[k-1]u[n-k]}_{1 \le k \le n}$$

$$= -6\left(\frac{1}{4}\right)^n \sum_{k=1}^{n} \left(\frac{4}{3}\right)^k =$$

same formula as in the previous convolution

$$= -6\left(\frac{1}{4}\right)^n \left(\frac{\frac{4}{3} - \left(\frac{4}{3}\right)^{n+1}}{1 - \frac{4}{3}}\right) =$$

$$= \left(\frac{1}{4}\right)^n \left(\frac{-8 + 6\left(\frac{4}{3}\right)^n \left(\frac{4}{3}\right)}{-\frac{1}{3}} \right) =$$

$$= 3\left(\frac{1}{4}\right)^n \left(8 - 8\left(\frac{4}{3}\right)^n \right) =$$

$$= 24\left(\frac{1}{4}\right)^n \left(1 - \left(\frac{4}{3}\right)^n \right) =$$

$$= 24\left(\frac{1}{4}\right)^n - 24\left(\frac{1}{3}\right)^n =$$

$$= 8\left(\frac{1}{3}\right)^{n-1} \left(\left(\frac{3}{4}\right)^n - 1 \right), \quad n \geq 1, \text{ so:}$$

$$\boxed{y_{zs}[n] = 8\left(\frac{1}{3}\right)^{n-1} \left(\left(\frac{3}{4}\right)^n - 1 \right) u[n-1]}$$

(d) Since the roots of its characteristic equation are all inside the unit circle, i.e., $|\lambda_i| < 1$, $\forall i$, then the system is BIBO stable.

(e) From our theory: $y_t[n] = y_{zi}[n] + y_{zs}[n] \Rightarrow$

$$\Rightarrow y_t[n] = 9\left(\frac{1}{4}\right)^n u[n] + 8\left(\frac{1}{3}\right)^{n-1} \left(\left(\frac{3}{4}\right)^n - 1 \right) u[n-1].$$

END.